**D 100186** **(Pages : 4)** **Name..................................**

**Reg. No...............................**

## SIXTH SEMESTER UG (CUCBCSS-UG) DEGREE EXAMINATION, MARCH 2024

Mathematics

MAT 6B 01—COMPLEX ANALYSIS

(2018 Admissions only)

Time : Three Hours Maximum Marks : 120

### Section A

*Answer* **all** *questions.*
*Each question carries 1 mark.*

1. For the complex numbers $z_0 = a + ib$ and $z_1 = c + id$, $\lim_{z \to z_0} z_1 =$ ________.

2. State True or False : The function

$$f(z) = \begin{cases} \dfrac{\operatorname{Re} z}{|z|}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

is continuous at $z = 0$.

3. Find the derivative of $f(z) = z^2$.

4. Compute the principal value of $\log_e z$ when $z = 1 + i$.

5. Find all the roots of the equation $\tan z = 1$.

6. Evaluate $\int_i^{1+4i} z^2 dz$.

7. If C is the simple closed contour given by the circle $|z| = 2$, then $\int_C dz =$ ________.

8. Integrate $\dfrac{z^2 + 1}{z^2 - 1}$ in the contour clock wise sense around a circle of radius 1 with centre at the point $z = \dfrac{1}{2}$.

9. State Maximum modulus principle.

10. If the sequence $\sqrt[n]{|a_n|}, n = 1, 2, \ldots$ converges with the limit L > 0, then the radius of convergence of the power series

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n = a_0 + a_1 (z - z_0) + a_2 (z - z_0)^2 + \ldots$$

is ________.

**Turn over**

11. Define removable singular point.

12. State Cauchy's Residue Theorem.

(12 × 1 = 12 marks)

## Section B

*Answer any* **ten** *questions.*
*Each question carries 4 marks.*

13. Prove that if the limit of the function $f(z)$ exists at a point $z_0$, then it is unique.

14. Verify Cauchy-Riemann equations for the function $f$ given by $f(z) = \dfrac{x - iy}{x^2 + y^2}$.

15. Examine the differentiability at the origin of the function $f$ given by $f(z) = |z|^2$.

16. Examine the analyticity of $f(z) = \cosh x \cos y + i \sinh x \sin y$.

17. Find the real part of $e^{-3z}$.

18. Prove that $\sinh(z_1 + z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_1$.

19. Evaluate $\int_0^1 (1 + it)^2 \, dt$.

20. If $f'(z) = 0$ everywhere in a domain D, then prove that $f(z)$ must be constant throughout D.

21. If M be any non-negative cosntant such that $|f(z)| \le M$ everywhere on a contour C and L is the length of C, then prove that $\left|\int_C f(z)\,dz\right| \le ML$.

22. Evalaute $\int_C \dfrac{dz}{z - a}$, where C is the circle $|z - a| = r$ oriented in the positive direction.

23. Examine the convergence of the series $\sum_{n=1}^{\infty} \dfrac{z^n}{n^2}$ on its circle of convergence.

24. Find the expansion for $f(z) = z^2 e^z$.

25. What kind of singularity the function $\dfrac{\cot \pi z}{(z - a)^2}$ has at the point $z = a$.

26. Find the residue of the function $f(z) = \dfrac{z}{z^4 + 4}$ at the isolated singular point $1 + i$.

(10 × 4 = 40 marks)

## Section C

*Answer any* **six** *questions.*
*Each question carries 7 marks.*

27. If $w = f(z) = \bar{z}$, show that $\frac{dw}{dz}$ does not exist at the origin.

28. Evaluate $\oint_C \frac{dz}{z^2+9} dz,$ where C is the unit circle.

30. Using principle of deformation of paths, evaluate $\int_C \frac{1}{z} dz$ where C is any positively oriented closed contour surrounding the origin.

31. Find an analytic function whose real part $e^x(x\cos y - y\sin y)$ and which takes the value $e$ at $z = 1$.

32. If $R_1$ and $R_2$ are the radius of convergences of the power series $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} b_n z^n$ respectively, show that the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n b_n z^n$ is $R_1 R_2$.

33. Find series representation of $f(z) = \frac{-1}{(z-1)(z-2)}$.

34. If C is a simple closed contour containing the origin, show that $\frac{1}{2\pi i}\int_C \frac{e^{az}}{z^{n+1}} dz = \frac{a^n}{n!}$.

35. Show that if $s > 0, a > 0,$ then prove that $\int_{-\infty}^{\infty} \frac{e^{1sx}}{x^2+a^2} dx = \frac{\pi}{a} e^{-as}$.

(6 × 7 = 42 marks)

## Section D

*Answer any* **two** *questions.*
*Each question carries 13 marks.*

36. (a) Find the Laurent series of $f(z) = \frac{1}{1-z^2}$ with centre at $z = 1$.

(b) If R be the radius of convergence of $\sum_{n=0}^{\infty} a_n z^n$ what is the radius of convergence of $\sum_{n=0}^{\infty} a_n^2 z^n$.

**Turn over**

37. (a) Evaluate $\oint_C \sec z\, dz$, where C is the unit circle.

(b) If $w = f(z) = \bar{z}$, show that $\frac{dw}{dz}$ doesn't exist at any point.

38. (a) Prove by contour integration that $\int_0^{2\pi} e^{\cos\theta} \cos(n\theta - \sin\theta)\, d\theta = \frac{2\pi}{n!}$ ($n$ positive integer).

(b) Show that $\int_{-\infty}^{\infty} \frac{\cos 3x}{\left(x^2+1\right)^2}\, dx = \frac{2\pi}{e^3}$.

(2 × 13 = 26 marks)