SOME CONTRIBUTIONS TO OPTIMAL RELIABILITY TEST PLANS AND ESTIMATION

A

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Bajeel P N

DECLARATION

"I hereby declare that this submission is my own work and that, to the best of my knowledge and belief, it contains no material previously published or written by another person nor material which has been accepted for the award of any other degree or diploma of the university or other institute of higher learning, except where due acknowledgment has been made in the text."

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CERTIFICATE

*This is to certify that the thesis entitled "**SOME CONTRIBUTIONS TO OPTIMAL RELIABILITY TEST PLANS AND ESTIMATION**" submitted by **Mr. Bajeel P N** to National Institute of Technology Calicut, for the award of the degree of the Doctor of Philosophy is a bonafide record of research work carried out by him under my guidance and supervision. The content of the thesis, in full or parts have not been submitted to any other Institute or University for the award of any other degree or diploma.*

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Abstract

Design of optimal system reliability test plan has numerous advantages in product manufacturing industry as it helps to evaluate the system reliability and thereby demonstrates that the system will perform satisfactorily, prior to its deployment to the concerned field. The failure data and prior information available on failure rates of units will help manufacturers to develop higher warranty periods and better service facilities to customers. Thus, a Bayesian approach can be adopted to obtain a good reliability estimates and cost-effective optimal test plans. In systems that are designed to achieve very high reliability (e.g., missiles, rockets, etc.), it is difficult to obtain good estimate of system reliability. This is due to either the data recorded contain only a small number of failures or lack of availability of sufficient testing time to observe failures. In such circumstances, a better alternative for obtaining data quickly is the application of accelerated life tests (ALTs) or partially accelerated life tests (PALTs). In both of these testing procedures, units are subjected to perform under higher stress than the normal stress level, whereas in PALT, some units are allowed to perform under normal stress. Both testing procedures are destructive in nature, i.e., one has to destroy some units to obtain lifetime quickly. Thus, in most of the existing reliability estimation method, failure time data are obtained from either classical life test or accelerated testing. In most of such situations, observing data is time-consuming, expensive or impracticable. One can make use of readily available degradation data enclosing information about system failure. For example, a fluorescent lamp is considered as a failed one, when its luminosity falls below a certain level, where an interesting feature of this experiment is the periodic monitoring of the luminosity. Hence degradation data, which are obtained during the lifetime of

units can be utilized to get reliability estimate saving time and money. This thesis focuses on the design of optimal reliability test plans for multi-component systems, and obtaining reliability estimates for systems using various techniques, such as using classical data with covariate information, ALTs, PALTs, and degradation data.

Initially, the thesis addresses estimation problems in a parallel system with n independent components. The lifetime of components in the system are assumed to follow exponential distribution with parameter $\boldsymbol{\lambda}$, where this $\boldsymbol{\lambda}$ is different for each component. In the literature, this $\boldsymbol{\lambda}$ is considered as constant while designing reliability test plans for parallel systems. But, in general, $\boldsymbol{\lambda}$ is not necessarily a constant; for example, the system performance may be affected by natural covariates such as temperature, pressure, and humidity. Thus, an attempt is made to construct reliability test plans for a parallel system, where the failure rate $\boldsymbol{\lambda}$ is considered as a function of covariates. An unbiased estimator and a maximum likelihood estimator for $\boldsymbol{\lambda}$ are obtained to construct system reliability estimate. A new strategy is adopted to replace the Acceptable reliability level (ARL) and Unacceptable reliability level (URL) by Acceptable reliability interval (ARI) and Unacceptable reliability interval (URI) respectively. The advantage of this strategy is that it reduces the burden of huge rejection cost as compared to that with traditional test plans. Several examples are discussed to illustrate the resulting test plans which lead to significant savings in testing costs.

The problem of designing reliability test plan for a series system is considered next. The focus is on designing component reliability test plans for a series system with n independent components. The advantage in this situation is that components can be tested at different locations, and finally, the system reliability test can be performed by incorporating individual component failure data. Based on data obtained from Type-II censoring, unbiased estimators for failure rates, and maximum likelihood estimator for system reliability are constructed. The design parameters are obtained by formulating an optimization problem which minimizes the maximum expected testing cost. It is observed that testing cost under Type-II censoring is random. To handle this random testing cost, an efficient algorithm is developed to minimize the total expected testing cost. Moreover, a simulation study is conducted to ensure that the derived sampling plan meets the specified producers and consumers risks requirements. Further, sensitivity analysis and qualitative analysis are made to study the effect of various input parameters and to discuss the nature of reliability acceptance sampling plans. It is noted that the developed test plan has the potential of reducing testing costs of about 80% in cost reduction compared to that in existing test plans. In addition to this, it is observed that about 70% reduction in the number of components to be tested for failure, as compared to that with respect to existing plans in the literature.

Bayesian statistical methods are becoming evermore popular in applied and fundamental research. Since abundant data on failure are available in the industry in the form of prior information, the design of Bayesian reliability test plans for systems is considered in this work. Series and parallel systems with n different components are studied. The lifetime of each component is assumed to follow an exponential distribution with unknown parameter λ. The prior information available on λ is modeled by Quasi-density function, and thereby a Bayes estimator is obtained for system reliability, based on data obtained from Type-I censoring. Examples are discussed to illustrate the resulting test plan that minimizes the total testing cost involved. It is noted that the Bayesian plan has about 70% savings in testing costs as compared to that with existing test plans in the literature.

In a typical life data analysis, the reliability practitioner analyses life data (time to failure) from samples of units operating under normal working conditions in order to quantify the life characteristics of the product and make predictions about all of the units in the population. For a variety of reasons, manufacturers wish to obtain reliability results more quickly than they can when the data comes from products operating under normal conditions. An alternative for this kind of situations is to use accelerated life tests to capture life data for products under accelerated stress conditions. In this line, a novel attempt is made to construct reliability acceptance sampling plans for Weibull distribution under constant-stress PALT. The required data for constructing sampling plans are obtained from Type-II censoring. Linear and Arrhenius stress relationships are used, and MLEs of Weibull parameters and acceleration factor are obtained. Further, exact distributions of some of the pivotal quantities involved in estimating parameters in linear and Arrhenius stress relations are obtained. Since the testing cost involved is random, an expression for expected total cost is given and thereby optimal sampling plans are obtained. Some examples are also discussed to illustrate the resulting sampling plan, and the testing costs are compared as well. It is observed that plan based on Arrhenius stress model has minimum testing cost as compared to plan based on linear stress model. Also, a sensitivity analysis is carried out to analyze the effect of a change in consumer's and producer's risks.

Finally, as a substitute to the destructive testing procedure in estimating system reliability, readily available degradation data of systems are considered. An exponential degradation path is considered with degradation rate parameter following Weibull distribution. Unknown scale parameter of Weibull distribution is estimated. The method of Bayesian estimation is also used to estimate the parameter and thereby system reliability, by considering informative (Gamma) and non-informative (Quasi) priors for scale parameter. The standard error for estimated scale parameter corresponding to both informative and non-informative priors are obtained using the Bootstrap method.

The various approaches to system reliability estimation and test plans discussed in this thesis are suitable for realistic situations and have an advantage of savings in testing costs.

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Chapter 1

Introduction and Some Preliminaries

1.1 Brief historical background

Reliability is a popular concept that has been celebrated as an admirable attribute of a person or a product. The English dictionary of Oxford defines it as "the quality of being reliable, that may be relied upon; in which reliance or confidence may be put; trustworthy, safe, and sure."The term 'reliability'was first coined by poet Samuel Taylor Coleridge [90]. Before world war II, reliability as a word meant dependability or repeatability. The modern use was re-introduced by the U.S. military in the 1940s and evolved to the present. In the early 1960s, increased specialization in reliability engineering, led to the consolidation of various efforts addressing reliability issues in components. Reliability engineering proceeded in two different tracks - first, a specialization in the discipline, second, there started growing trend from component level reliability to system based attributes, keeping in view of system performance and availability. Then, there comes to the concept of reliability test plans to test the reliability of system or component failures over a specified period.

An early application of reliability is related to the telegraph which was a system powered by a battery with simple transmitters and receivers connected by a wire. The main failure was due to either a damaged wire or insufficient voltage. By 1915, radios with a few vacuum tubes began to appear throughout the world. Automobiles were commonly used by 1920 and represented mechanical applications of reliability.

In the 1920s product improvement through the use of statistical quality control was initiated by Dr. Walter A. Shewhart at Bell Labs [50]. In the twentieth century, on a parallel path with product reliability, the development of statistics started. Statistics as a tool for making measurements soon became inseparable with the development of reliability concept. For a decade, between 1920 and 1930, Taylor worked on ways to make products more consistent and the manufacturing process more efficient. He was the first to be responsible for separating the engineering from management and control [52]. During this period, Wallodie Weibull investigated the fatigue of materials in Sweden. During this time, he created a distribution, which is presently called as Weibull. The world war II gradually gave a way to cost considerations for the military. Nearly half of their essential types of equipment were non-functional all of the time which resulted in a huge financial burden. IEEE formed the Reliability Society in 1948 to redress the grievances. In-order to study reliability issues with the air force, Rome Air Development Center (RADC) was established in Rome (1951). By 1959, Wallodie Weibull published "Statistical Evaluation of Data from Fatigue and Creep Rupture Tests: Fundamental Concepts and General Methods"as a report for the U. S military.

On the military side, Advisory Group on the Reliability of Electronic Equipment (AGREE) was formed in 1950. By 1952, an initial report suggesting, (a) need to develop better components and more consistency from suppliers (b) the military should establish quality and reliability requirements for component suppliers (c) actual failure (field) data must be collected on components in order to identify the causes of problems, for the creation of reliable systems, was published. In 1957, a final report was generated by the AGREE suggesting some modifications like developing replaceable Standard Electronic Modules to restore a failed system, running formal demonstration tests with statistical confidence for products, running longer and harsher environmental tests that include temperature extremes and vibrations etc. [19]. Over the next several decades, Birnbaum made significant contribution in probabilistic inequalities, nonparametric statistics, reliability of complex systems etc.

In 1960s, a strong commitment to space exploration made NASA, a driving force for improved reliability of components on systems. During this decade, people started using and contributing to the growth and development of the Weibull function/Weibull graph, and the propagation of Weibull analysis methods and applications. In 1962, G.A. Dodson and B.T. Howard of Bell Labs published "High-Stress Aging to Failure of Semiconductor Devices"in the Proceedings of the $7-th$ National Symposium of Reliability and Quality Control, justifying the Arrhenius model for semiconductors. During the 1970s, the use and variety of ICs increased. Bipolar, NMOS and CMOS, etc. are developed amazingly passive components, which were once covered by IRPS, moved to a Capacitor and Resistor Technology Symposium. By the end of the decade, the Navy Material Command brought in Willis Willoughby from NASA to help in improving military reliability across a variety of platforms. He had been responsible for making sure the Apollo spacecraft worked reliably all the way to the moon and back.

During the decade of 1980s, rapid changes started coming in. Televisions composed of semiconductors, automobiles rapidly used semiconductors with various microcomputers. Communication systems began to adopt electronics instead of older mechanical switching systems. By 1990s, the place of integrated circuits (ICs) started picking up. Wider use of stand-alone microcomputers was increased and, the military reliability changed to a different dimension enabling new approaches and measures. On the software side, the capability maturity model (CMM) was generated, and the companies with the highest levels of the model were thought to have less residual faults.

New approaches were required such as software mirroring, rolling upgrades, hot swapping, self-healing, and architecture changes [62]. ISO 9000 added reliability measures of the development and design portion of the certification. The turn of the century saw the expansion of the world-wide-web which created new challenges of security and trust. The older problem of little reliability information available had changed to too much information of questionable value. New technologies like webinars, net banking, micro-electro mechanical systems, hand-held GPS, hand-held devices which are the combination of cell phones and computers, all represent challenges to maintain reliability. In many ways, reliability has become the part of everyday life of human beings and consumer expectations (see [63]).

1.2 Motivation and review of literature

Reliability and life testings are very fast growing and vital fields in consumer and capital goods industries, in the space and defense industries, and in NASA. Both of them provide the theoretical and practical tools by which it can be assured that the probability and capability of parts, products, systems, and components to perform their required functions in definite environments for the desired period without failure. It is necessary for any industry to be competitive regarding technology in view of liberalization and globalization. Thus, all industries have to know the reliability of their products, have to be able to control it and have to produce them at the optimum reliability level that yields the minimum life-cycle cost to the customer. Designing and developing a product which satisfies the requirements of the performance specifications during a relatively short period of the input-output and efficiency of performance tests, but fails shortly thereafter or before the required period of function or mission time, is of no use. Reliability engineers and practitioners should be well aware of new methods of reliability and life testing in-order to optimally minimize product failures.

The most important statistical method of assessing the quality of products is the reliability measure. As consumers in the market care a lot about quality characteristics, the design of optimal reliability test plans, which is less time-consuming and cost-effective, help the producer to test the reliability of their products and release them into the market and earn good profits. Currently, available reliability test designs are not very much cost effective and, need further attention for improvements in design. The estimates of reliability used by many industries are rudimentary or not supported by proper statistical theory. Taking these factors into accounts, this research work is focused on contributing to the design of reliability test plans by using theoretical statistical approach, and thereby yielding highly cost-effective component reliability test plans.

Kececioglu, in his book 'Reliability & Life Testing Handbook' [54] explained the objectives of reliability and life testing. Reliability and life testing determine if the performance of components, equipment, and systems, either under closely controlled and known stress conditions in a testing laboratory or under field use conditions, with or without corrective and preventive maintenance, and with known operating procedures, is within specifications for the desired function period, and if it is not, whether it is the result of a malfunction or of a failure which requires corrective action. Determining the pattern of recurring failures, the causes of failure, the underlying times-to-failure distribution, the associated stress levels, the failure rate, the mean life, and the reliability of components, equipment, and systems and their associated confidence limits at desired confidence levels are noted as the objectives of the test. Reliability and life testing provide guidelines as to actions to be taken based on the results obtained. Kececioglu further states that appropriate procedure in life testing which provide reevaluation of the reliability wise performance of the units after corrective actions are taken to provide a means to determine, with chosen risks, whether a redesign has indeed improved the failure rate, mean life, or reliability of components or equipment with the desired confidence, to provide a statistical means to determine which one of two manufactures should be preferred. The author clearly explains various elements and suggestions about the reliability and life testing in this book.

1.2.1 System based component reliability test plans

It must be noted that reliability and life tests at the component level are different from that at the system level. A component has more precisely definable function in a system. A system is a major functional unit, may be composed of a multitude of interacting components or independent components. A component has to be much more reliable than the system in which it is functioning; consequently, to determine its failure rate, mean life or reliability, a large sample size is required, a large number of tests need to be conducted over a wide range of conditions that include burn-in tests and simulate system environments, and over a relatively long period of time, to determine the best range of its use in the system it is destined for and to demonstrate its reliability.

The expected lifetime and failure rate of units or products are of great significant factors for any manufacturing industry. If a statistically significant sample is selected randomly from a lot of similar units and if the selected units are tested till all units fail, then enough data will get generated for confidently predicting the life expectancy of any unit contained in the lot. Life testing is carried out for estimating the mean time to failure and expected life of units. When units are produced in the manufacturing industry, the manufacturer would like to study the exact behavior of units in the normal working environment. For that, he needs data and evaluates life and reliability of the units produced. By studying this, he can improve the design, manufacturing, and other related goals. This will improve the confidence of the manufacturer and thus to provide higher guarantee periods and better service facilities. Also, he can plan maintenance schedules, replacements of parts, warranty policies, etc.

A dedicated program of testing is an important ingredient in the development of a complex system. The usual objective is to evaluate the reliability of a system and demonstrate that it will perform satisfactorily, prior to actual deployment in the field. To derive optimum component test plans, the problem can be formulated based on the notion of producer's and consumer's risks in acceptance sampling plans. Formally, producer's risk is defined as the probability that an acceptable system may be erroneously rejected by the test plan, whereas consumer's risk is defined as the probability that an unacceptable system may be erroneously accepted by the plan. The errors associated with producer's and consumer's risks are respectively called the Type-I and Type-II errors (see, [82]). The main step of development of the highly reliable system is the testing and analysis of the system. The purpose of this testing and analyzing is to demonstrate that the system under consideration will perform at a level of reliability that is acceptable concerning the mission for which it is designed. Before the actual production, put the system for testing and evaluate the reliability. The reason for doing this is to ensure that the system will achieve some predefined reliability requirements. Analogous to standard acceptance sampling plans, reliability test plans can be expressed in terms of maximum acceptable values for producer's risk (Type-I error) and consumer's risk (Type-II error). System level tests tend to be very highly scrutinized because of the length of time and the cost that they add to the overall design and development schedule; however, they are often unavoidable. On the other hand, there are situations where a better approach might be to use component test plans for system reliability demonstration (see, [83]). There are numerous advantages to this approach: it is less expensive and less time taking, testing can proceed at different times and locations, test facilities for components tend to be simpler and less expensive, and it is more cost effective since the entire system is not assembled until it is guaranteed to be reliable. Several variations of the two basic options (component level and system level) are possible. For instance, one may conduct tests at the level of sub-assemblies (or subsystems) that are composed of individual components, and that are themselves part of the entire system. Alternatively, one could adopt some combination of system, subsystem, and component testing. The choice of an appropriate test plan depends on the characteristics of the specific system under consideration, as well as cost and feasibility. System level tests are typically more expensive as well as more inconvenient than component level tests. There are several reasons for this: (a) The instrumentation, test fixtures and facilities, materials and the actual test units all usually cost more for systems than for components. (b) With component tests, the entire system need not be assembled prior to testing. This results in significant savings both concerning costs as well as the times associated with system development. (c) Testing may proceed independently at different times and locations. This is an important issue since the components (or subsystems) of many large systems are typically contracted to different organizations who develop these components independently. (d) Component tests tend to be more informative in that the behavior of individual components can often be understood better than in the case where they are assembled into a larger system. (e) Finally, component testing goes well with the notions of total quality since the final system is assembled only after a performance guarantee is obtained.

Conversely, system testing is preferable when the component failures are highly dependent on each other since component testing assumes independence of component failure times and the availability of a mathematical model that expresses system reliability in terms of component reliabilities. Another situation where system testing may be preferable is when the interfaces between components are inherently unreliable; however, in such situations, it may still be more economical to use a combination of system and component testing. (See, [83, 84]).

The basic problem was first considered by Gal (1974: [36]). In this paper, he studied a situation where a certain unacceptable reliability level, R_0, needs to be demonstrated at specified confidence $1-\alpha$. He assumed exponential life distributions for components and derived a general solution procedure to compute the optimum component test times which minimize the total test cost while guaranteeing the probability requirement

$$P(\text{Accept the system} \mid \text{System is bad}) \leq \alpha.$$

The system is accepted if and only if there is no component failure during the test. Mazumdar (1977: [59]) extended the model, that was presented by Gal (1974: [36]), to the situation where in addition to an unacceptable system reliability level, certain acceptable reliability level needs to be demonstrated at specific confidence $1-\beta$. He added the constraint

$$P(\text{Reject the system} \mid \text{System is good}) \leq \beta.$$

In the context of quality control, the significance levels α and β are also known respectively as consumer's and producer's risks. Mazumdar (1977: [59]) used the following decision rule to accept a system: "Test (with replacement) each component j for a total of t_j time units. Observe the number of failures for each component. If the total number of failures is less than a pre-assigned number, accept the system; otherwise, reject it". This rule for accepting a system is referred to as the sum rule in the sequel. Using the sum rule, Mazumdar (1977: [59]) gave a solution for the optimum component test times for a series system. Within the framework of their formulations, both Gal (1974: [36]) and Mazumdar (1977: [59]) showed that for a series system, the optimum component test times are independent of component test costs, and are identical. In all this work, it was assumed that no prior information is available about component reliabilities.

Altinel (1992: [14]) studied this problem first for a series system of k components when upper bounds on individual component failures are given as the prior information; he has shown that optimum component test times are not identical, and the use of such prior information also leads to reduced total test costs. Note that in this work, the upper bound considered is a pure constant and not a function of any covariates that affect the functioning of the component or system. He also developed a procedure to compute optimum component test times. Altinel (1994: [9]) extended these results for a problem which is more general in two aspects. First of all, the only assumption he made on the system reliability function is that it is an increasing function of component reliabilities, or equivalently a decreasing function of component failure rates, which is true for all coherent systems. Also, the deterministic prior information on component failure rates can restrict their selection to any compact and non-empty subset of non-negative real numbers. This subset does not necessarily have to be in the form of simple upper bounds on the failure rates, which limits each of them with a real closed interval; it can be a polyhedral set, or a hyper-sphere, etc.

Different aspects of the test design problem for a series system have been addressed in [14, 60, 81]. Rajgopal and Mazumdar (1995: [81]) considered a series system consisting of components with unequal test costs and unknown constant failure rates with pre-assigned bounds on the failure rate. They derived minimum cost component test plans that guarantee specified values of producer's and consumer's risks on the system reliability. These plans are derived for the case in which the component failure times are exponentially distributed, as well as for the case in which they follow a Gamma distribution. The optimization problems are formulated and solved via standard mathematical programming techniques. Mazumdar (1980: [60]) considered the problem of optimizing component testing for a series system with redundant subsystems so as to minimize the total cost of testing with given constraints as, the probability of accepting the system must be less than α if the system is not good (that is, the reliability of the system is less than a pre-assigned unacceptable level), and the probability of rejecting the system must be less than β if the system is good (that is, the reliability of the system is greater than a pre-assigned acceptable level). This problem was solved previously for a series system under the sum rule acceptance criterion. This result is extended in the paper [60] to cover the more general case of a series system that has built-in redundancy in that it consists of subsystems each of which contains several identical components in parallel. The optimum testing time obtained is shown to be independent of the individual component testing costs. Altinel (1992: [14]) developed an acceptance procedure for a series system based on component tests that guarantee certain probability requirements on Type-I and Type-II errors. Methods for computing the individual component test times which minimize total test cost and guarantee the probability requirements for this acceptance procedure have also been given. In the paper [14], a priori deterministic information on individual component failure rates is available. Results have shown that it is profitable to use the given information, and optimum component test times are not necessarily equal: they depend on individual component test costs. They provide a procedure to compute optimum component test times. This procedure is based on the well-known cutting plane idea and column generation technique. They also discuss the conditions for the existence of an optimum solution. Rajgopal, Mazumdar, and Savits (1994: [85]) derive some of the properties of the Poisson distribution that are not commonly known or used to construct certain class of component tests for verification of a series system reliability.

Parallel systems have also been addressed in [100] and [79]. Yan and Mazumdar (1987: [100]) considered a parallel system of n independent components with constant failure rates and the component testing procedure guarantee that the given consumer and producer risks are not exceeded. They give certain restrictions on the magnitude of the unknown failure rates for guaranteeing the producer risk. The component test procedures use Type-I censoring and use decision rules based on (a) the total number of component failures during the testing periods, (b) the number of failures for each component, and (c) the maximum likelihood estimate of system reliability. Observe that among these decision rules, (a) and (b) do not focus on the importance of taking system reliability estimate into consideration for designing test plans. Rajgopal and Mazumdar (1988: [79]) considered the problem of acceptance test plans for a parallel system of different components with constant failure rates. The components are individually tested, and the tests are terminated as soon as a pre-assigned number of components fail. This paper provides a criterion for accepting or rejecting the system based on the sum of the logarithms of the total times on test for each component. The critical level for the test statistic is chosen to guarantee that the specified consumer and producer risks on the system reliability are not exceeded. The use of this statistic makes the computation of these critical values much simpler as compared with that of a previously used statistic based on the product of the total times on test for each component. Several approximate procedures are considered for deriving these critical values. The paper also formulates the optimization problem for deriving the minimum cost component testing plans when a Type-II censored component-test procedure is used for a parallel system with n independent components.

Rajgopal and Mazumdar (1996: [82]) considered the problem of acceptance test plan for a series system of n different components, each having an unknown, but a constant failure rate. Components are individually tested, and the tests are terminated when a pre-assigned number of failures are observed for each component. The total time on test for each component is noted, and a statistic is constructed by using the observed test times and the number of failures of different components. The statistic was based on an MLE of system reliability. This statistic based on component test times is then used in specifying a decision rule for accepting or rejecting the entire system. The design of the test plan is stated as an optimization problem which minimizes test costs while ensuring that specified consumer and producer risks on the system reliability are not exceeded. In this work, they have shown that all components in a series system have to be tested equally. But, in practice, this may not be true. Moreover, the total test cost considered in their work is not realistic, since cost under Type-II censoring is not necessarily a constant quantity.

Altinel, Ozekici, and Feyzioglu (2001: [12]) considered the component testing problem of a series system with redundant subsystems, where all components fail exponentially. In this work, the component failure rates are not constant parameters, but they change dynamically with respect to time. Also, the optimal component testing problem is formulated as a semi-infinite linear programming problem. They also presented an algorithmic procedure to compute optimal test times based on the column generation technique. Altinel, Ozekici, and Feyzioglu (2002: [13]) considered the component testing of a series system in a random mission. In this paper, they have shown that the realistic model can be handled using available results in semi-infinite linear programming and difference of convex functions programming.

Vellaisamy and Kumar (2008: [72]) constructed a reliability test plan for a parallel system with n independent components under Type-II censoring with the assumption that the lifetime of each component follows an exponential distribution with a constant but unknown failure rate parameter. They derived optimal reliability test plans which ensure the usual probability requirements on system reliability. Further, they solved the associated nonlinear integer programming problem by a simple enumeration of integers over the feasible range and developed an algorithm to obtain integer solutions with minimum cost. Vellaisamy and Kumar (2010: [73]) constructed a reliability test plan for a series system with n independent components under mixed censoring, a combination of both Type-I and Type-II censoring, with the assumption that the lifetime of each component follows an exponential distribution with a constant but unknown failure rate parameter. In their work, they have shown that optimal sample size is the same for all components.

Most of the component level reliability test plans that are available in literature have been developed for series and parallel systems under the assumption that components have constant failure rates. Sabnis and Agnihothram (2006: [88]) construct a reliability test plan for a parallel system with failure rates of the corresponding components depending upon covariates. With the motivation of developing more realistic test plans Sabnis and Agnihothram (2007: [89]) construct reliability test plans for a series systems under the assumptions that the component lifetimes are independently distributed exponential random variables. The failure rates of these exponential random variables depend on covariates. In their work, they obtain data from Type-I censoring. However, the reliability test plans for a parallel system with n different components in the presence of covariates is not addressed in the literature, by using data from Type-II censoring. Similarly, reliability test plan for a series system in the presence of covariates, by using data from Type-II censoring is not considered yet.

The initial contributions in this thesis is to construct optimal reliability test plan for parallel and series systems having different components with independently distributed exponential lifetimes, and by obtaining data from Type-II censoring, while considering covariate information available on failure rates. More meaningful test plans are obtained by computing system reliability estimates, and by using upper bound on failure rates as a function of covariates. In addition to this, a random testing cost under Type-II censoring is considered, and the corresponding optimization problem is solved by using the expression for maximum-total-expected-testing-cost, for handling the problem of testing reliability for a series system. Thus an attempt is made in this thesis to address the issue of obtaining optimal reliability test plan in more realistic situations, by obtaining data from Type-II censoring. Next, using prior information available on failure rates of components in the system, an attempt is made to construct Bayesian reliability test plans for series/parallel systems by obtaining data from Type-I censoring. This Bayesian plan has numerous advantages over available classical plans, in terms of savings in testing costs.

1.2.2 Partially accelerated life test and acceptance sampling plans

In life testing, acquiring life test data at a specified normal use condition require a long period. This problem makes life testing a difficult, time consuming and a costly procedure. Under such circumstances accelerated life tests (ALTs) or partially accelerated life tests (PALTs), which shorten the lives of test units are used. ALT and PALT differ on the conditions at which they are applied. The test units are run only at accelerated conditions in an ALT, whereas test units are run both at accelerated and normal use conditions in PALT.

Usage of ALT can often be seen in reliability prediction. Here, in order to induce early failures, specimens are tested at high-stress levels. Then, through an existing stress dependent model, the failure information is related to specimens at an operational stress level. In the absence of such a model, the ALT can't be conducted. In such conditions PALT, which is a combination of both ordinary and accelerated life tests, becomes a suitable option. Reliability analysis by the application of PALT helps to save time and money as compared to that in ordinary or traditional life tests.

Chernoff (1962: [23]) and Bessler et al. (1962: [87]) coined and studied the concept of accelerated life tests. The parameter θ in these tests, which appears in the underlying distribution function, is considered as a specified function say $\theta = \phi(s,\ \eta)$. Here, s is an environmental stress to which an unit on test can be subjected, and η is an unknown parameter, which needs to be estimated from data. They consider the problems of estimation of unknown parameter and of the optimal design of testing process in both sequential and non-sequential contexts. Lifetime distribution is assumed to be exponential, and ϕ is considered as linear function. Several authors, for example, [77], [25], [38], [1–3], [66, 67], [4], [49], [61], [7], [5] have studied PALT under Type-I and Type-II censoring. The time at which a test unit can be switched from the standard stress conditions to higher stress is under the control of the experimenter. This is the assumption made by DeGroot and Goel (1979: [25]). They also assumed that, in accelerated life testing, it will be possible to choose various levels of higher stresses. Therefore, they restricted themselves to problems in which higher stress levels were fixed in advance.

Optimal designs of partially accelerated life tests (PALTs) for Exponential distribution is considered in [27]. The sample proportion allocated to accelerated condition for the constant PALT is determined to minimize the asymptotic variance of MLE of the acceleration factor. They obtained optimal acceptance sampling plan under PALT, by minimizing the generalized asymptotic variance of estimators of failure rate and acceleration factor. However, no importance is given to minimize Type-I and Type-II error while designing the required acceptance sampling plans. Bai, Chung and Chun (1993: [15]) extended Bai and Chung (1992: [27]) to units having lognormally distributed lives. Assuming Weibull distribution as a lifetime model, the paper [48] considers optimum plans for failure-step stress partially accelerated life tests with two stress levels under Type-II censoring. The optimum proportion of test units failing at each stress according to a certain optimality criterion is determined by the optimum test plans. Here the D-optimality criterion is considered, and some numerical illustrations are provided for illustrating the proposed procedure.

Aly and Ismail (2008: [43]) discuss time-step stress partially accelerated life tests (PALT). Under Type-I censoring, the maximum likelihood estimation of the parameters of simple (only two stresses) time-step stress model is presented. Further, the confidence intervals of the estimators are constructed. Also, optimum time-step stress test plans are obtained. The optimal stress switching point is obtained from the optimum test plan. These plans minimize the generalized asymptotic variance of the maximum likelihood estimators for the model parameters. The test units generally used to follow Weibull lifetime distribution.

The paper [42] concerns with constant-stress partially accelerated life test with multiply censored data. In this paper, the lifetime of the test unit is assumed to follow an inverted Weibull distribution, and maximum likelihood estimates are obtained for the model parameters and the acceleration factor. The confidence intervals of the unknown parameters and acceleration factor are constructed for large sample sizes.

In the paper [78], an optimum design of constant-stress partially accelerated life test (PALT) plan is presented. The authors in [78] assumed that the product life follows Truncated logistic distribution truncated at point zero under Type-I censoring. In this paper, the optimal sample proportion allocated to both normal use condition and accelerated condition for the constant PALT is determined by minimizing the generalized asymptotic variance of MLEs of the acceleration factor and model parameters. The paper [45] deals with simple constant-stress Partially Accelerated Life Tests (PALT) with Type-II censoring. It is assumed in this paper that the lifetime at design stress has a Weibull distribution. They developed statistically optimal PALT plans such that the Generalized Asymptotic Variance (GAV) of the maximum likelihood estimators (MLEs) of the model parameters at design stress is minimized. The study in paper [44] considers constant-stress partially accelerated life tests for censored lifetime data, where the lifetime distribution is assumed to follow a log-logistic distribution.

The paper [47] discusses the estimation of Weibull distribution parameters based on hybrid censored data under constant-stress partially accelerated test model. Two estimation methods; maximum likelihood and percentile bootstrap are used to make statistical inference on the Weibull distribution parameters and the acceleration factor. The mean square errors of the estimators are calculated to evaluate their performances, through a Monte Carlo simulation study.

The constant stress partially accelerated life tests with Type-I censoring under Weibull distribution is considered in [46]. The maximum likelihood estimators of the model parameters are derived in this paper, and partially accelerated life test plans are developed such that the generalized asymptotic variance of the maximum likelihood estimators of the model parameters is minimized. In the paper [94], authors estimated the parameters of Weibull distribution in step-stress partially accelerated life tests under multiply censored data. The maximum likelihood estimates are used to obtain the parameters of the Weibull distribution and the acceleration factor under multiply censored data. Also, they obtained the confidence intervals for the estimators. The case of constant-stress partially accelerated life tests under Type-I censoring, with test units following Gompertz distribution, is discussed in [57]. Maximum likelihood estimates of parameters of distribution and acceleration factor are obtained. In addition, approximate confidence intervals of the parameters are also constructed, and optimum plans are obtained.

All the acceptance sampling plans discussed under constant-stress PALT obtained optimal plan parameters without considering the notion of minimizing testing costs subjected to the requirements of satisfying the probability of Type-I and Type-II error constraints. As the next contribution to this thesis, an attempt is made to construct an acceptance sampling plan for Weibull distribution using constant-stress partially accelerated life test. The optimal plan parameters are obtained by minimizing the total *expected testing cost subjected to the requirements of satisfying the probability of Type-I and Type-II error constraints. Two different life-stress relations are considered, and corresponding optimal test plans are compared.*

1.2.3 Degradation growth models and estimation

It is difficult to estimate the system reliability for systems that are designed to achieve high reliability using data that consists of a small number of failures obtained from life tests that record only time to failure. Reliability depends on the dynamic balance between stress which accumulates over time and strength for many such systems. For example, a vehicle axle fails when the depth of a crack exceeds a critical level (see [70]). Measurements taken over time on degradation or accumulated stress contain information about the reliability of the system concerned. Even with data from a relatively small number of units, one can hope to achieve better specification of reliability by harnessing this information (see [20]). Similar studies in this area were done in [93], [24], [22], [98], [96] and [97].

Murray (1993: [69]) initially performed the degradation data analysis by setting models to sample path for the individual units and obtained pseudo failure times. Further, these failure times were analyzed using common life data analysis methods. Meeker et.al (2009: [64]) took the random effects model to describe the unit-to-unit variability and that showed how a degradation model along with a failure definition, induces a failure time distribution. Marta. A. Freitas et al. (2009: [33]) made it clear by presenting three classical methods: namely, analytical, numerical and approximate methods to estimate the failure time distribution of degradation models $D(t) = \beta t$ and $D(t) = (1/\beta)t$. They used various parametric distributions such as Weibull, Lognormal and normal distributions for the random parameter. Illustrations were also presented as a case study on train wheel degradation data. Julio C. Fereira et.al (2012: [32]) discussed the case study on train wheel degradation data by taking $D(t) = \alpha_0 + e^{\eta}t$ as degradation model with η specifying the random effect parameter. Time to failure distribution of wheels is obtained based on the position of wheels. Later, Freitas et al. (2010: [34]) conducted a similar study by presenting five methods of degradation data analysis. Parametric degradation models by considering linear degradation paths and simple non-linear degradation paths, are studied by authors in [8], [58] and [40]. The non-linearity nature (increasing/decreasing) of measurements can be seen in real life examples like train wheel degradation data and drug potency degradation data.

In obtaining data from degradation measurements, systems are inspected at prefixed time points in designed experiments, and the current status of the systems are recorded along with measurements of accumulated stress. Failure can be defined for such systems in terms of a specified level of strength $s(t)$ at time t, and the reliability at the same instant is given by $R(t) = Pr[X(t) < s(t)]$, where $X(t)$ denotes the accumulated stress at the same instant (see, [17]).

There are several works on the modeling of degradation leading to failure. Gorjian et.al (2010: [39]) and Nikulin et.al (2010: [68]) are examples of the papers in this direction. In many of these models it is common to choose a fixed threshold $s(t) = s$ for the degradation $X(t)$ for all $t \geq 0$. The reliability at time t for these models is given by $R(t) = P[X(t) < s]$.

Degradation was modeled as a function of time t by Lu and Meeker (1993: [20]). The function is given by $X(t) = \mu(t, \bar{\theta}, \bar{\phi}),\ t \geq 0$, where $\bar{\phi}$ is a vector of fixed effect parameters, $\bar{\theta}$ is a vector of random effect parameters and the degradation is measured with additive error at specified times. In their model, the event of a critical crack length exceeding a constant level of 1.6 inches is defined as a failure (that is, $s(t) = 1.6$ inches). They had considered a data set consisting of fatigue crack length measurements at equi-spaced time points for many metallic specimens under test.

The function $X(t)$ is regarded as an observable without error and has been modeled as a Gamma process with the scale parameter explained by a random effect (see, [56]). For computational ease, they had made a simplifying assumption which resulted in their analysis exhibiting a lack of fit with experimental data. Park and Padgett (2005: [76]) considered the same data as that of [56] and modeled $X(t)$ alternatively as a Gamma process as well as geometric Brownian motion. The heterogeneity present among the degradation paths was however ignored. They assumed a specific parametric model for the degradation path to estimate the reliability function

$R(t)$ and developed a two-stage estimation methodology for the associated parameters. In general, any parametric methodology is sensitive to the assumed model, and hence the resulting estimator may be biased whenever the parametric form is incorrectly chosen.

Thus the degradation path is known for some systems, and in such cases, useful information on the reliability of a product can be obtained from these degradation measurements. As a contribution to estimation using degradation data, in this thesis, a degradation model having exponential degradation path with positive degradation rate, which follows a Weibull distribution with known shape parameter and unknown scale parameter is considered. The corresponding unknown parameters are estimated. Baye's estimate of scale parameter of Weibull distribution is also obtained, and thereby Bayesian reliability of first kind and second kind for the system are computed.

1.3 Some preliminaries

1.3.1 Concepts of reliability

Reliability provides the relationship between the age of a unit and the probability that the unit survives up to that age while starting the mission at age zero. The reliability function enables the determination of the conditional reliability function, the probability density function, the failure rate function and the mean life function. Reliability is the conditional probability, at a given confidence level, that the equipment will perform their intended functions satisfactorily or without failure; i.e., within specified performance limits, at a given age, for a specified length of time, function period or mission time, when used in the manner and for the purpose intended while operating under the specified application and operation environments with their associated stress levels.

Let T be the lifetime of a device, then t be the observed time to failure. Since T is a random variable, there is a probability distribution function of T, then

$$F(t) = P(T \leq t),\ 0 < t.$$

It represents the probability of failure in the interval [0, t]. Then the corresponding reliability function is given by

$$R(t) = P(T > t) = 1 - F(t) = \int_t^\infty f(x)dx,$$

where $f(t),\ t \in (0,\ \infty)$ is the probability density function of T.

Thus, reliability is the probability that the time to failure is equal to, or greater than, the mission duration. Then the unit cannot fail before the mission is completed, because it has operated for a time equal to or longer than the mission duration, and the probability of not failing before the mission is completed, is the reliability of the unit for that mission. In other words, if T represents the lifetime of a component, then the reliability at a time t is the probability that the life time T exceeds t. Note that $R(0) = 1\ \ and\ \ R(\infty) = 0$.

1.3.2 Conditional reliability

The conditional reliability is defined as the probability that a component or system will operate without failure for a mission time t given that it has already survived for a time T.

$$R(t/T) = \frac{R(t+T)}{R(T)}.$$

1.3.3 Failure rate function

The failure rate function, $\lambda(T)$, provides the relationship between the age of a unit and the failure frequency, or the number of failures occurring per unit time at age T. The failure rate function enables the determination of the reliability bath-tub curve. The failure rate $\lambda(t)$ of an item exhibiting a continuous failure-free operating time T is defined as

$$\lambda(t) = \lim_{\delta t \to 0} \frac{P(t < T < t + \delta t / T > t)}{\delta t} = \frac{f(t)}{1 - F(t)} = -\frac{R^{'}(t)}{R(t)}. \tag{1.3.1}$$

It is to be noted that $f(t)\delta t$, for small δt is the unconditional probability for failure in $(t, t + \delta t]$, given the item is new at $t = 0$. $\lambda(t)\delta t$ is the conditional probability that the item will fail in the interval $(t, t + \delta t]$ given that the item was new at t=0 and has not failed in $(0, t]$.

Assuming $R(0) = 1$, integration of equation (1.3.1) yields

$$R(t) = e^{-\int_0^t \lambda(x)dx}.$$

Example: If the failure distribution function follows an Exponential distribution with parameter λ , then the failure rate function is

$$\lambda(t) = \frac{f(t)}{R(t)} = \frac{\lambda\, exp(-\lambda t)}{exp(-\lambda t)} = \lambda.$$

This means that the failure rate function of the exponential distribution is a constant. In this case, the system does not have any aging property. This assumption is usually valid for software systems. However, for hardware systems, the failure rate could have other shapes.

1.3.3.1 Mean time to failure (MTTF)

The mean time to failure is the average, or the expected time to failure of identical units operating under identical application and operation environment stresses. Let T be the failure free operating time, then the mean of T is given by

$$MTTF = E(T) = \int_0^\infty tf(t)dt = \int_0^\infty R(t)dt.$$

Example: If the lifetime distribution function follows an exponential distribution with parameter λ, that is, $F(t) = 1 - exp(-\lambda t)$, the MTTF is

$$MTTF = \int\limits_0^\infty R(t)dt = \int\limits_0^\infty exp(\lambda t)dt = \frac{1}{\lambda}.$$

This is an important result as for exponential distribution. MTTF is related to a single model parameter in this case. Hence, if MTTF is known, the distribution is specified.

1.3.4 System reliability

The main concern of a system engineer is to estimate various reliability parameters of systems he is dealing with. The system may vary from simple to complex. One approach for analyzing such systems is to decompose them into subsystems of convenient size, each representing a specific function. Reliabilities of subsystems are then estimated and combined to determine the reliability of the entire system using certain probability laws. This approach requires a complete knowledge of the physical structure of the system and the nature of its functions sufficiently well to determine the behavior of the system in the event of failure of a subsystem. The subsystem may consist of one or more components whose reliabilities are known.

1.3.4.1 Systems with components in series

Consider a system having a total of n independent components. If the functional diagram suggests that the successful operation of the system depends upon the proper operation of all the n components, then we say that the system configuration is a series or chain type. The information at the IN end will reach the OUT end only if all the n components functions satisfactorily. Many complex systems can be reduced to such a simple structure. Let $p_i(t)$ is the probability that the component i is good at time t, then the time-dependent reliability function of the system is

$$R(t) = p_1(t)p_2(t)...p_n(t) = \prod_{i=1}^{n} p_i(t).$$

The reliability of a series system is always worse than the poorest component.

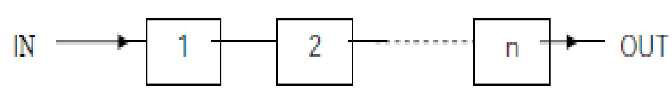

Figure 1.1: Series system

Example: Consider a series system having 3 components with reliabilities $R_1 = 0.6, R_2 = 0.9, R_3 = 0.8$, then the reliability of the system $R = R_1R_2R_3 = 0.432$.

1.3.4.2 Systems with components in parallel

A system with m components is known as an m- unit parallel system if and only if the successful functioning of any one of the components leads to the system success. The components are connected across each other and there are m parallel paths between the IN end and the OUT end and the existence of any of them is sufficient to transmit information from the IN end to the OUT end. In other words, The system fails only when all units fail. The system reliability is

$$R(t) = 1 - \prod_{i=1}^{m}(1 - p_i(t)).$$

The reliability of the parallel system increases with components.

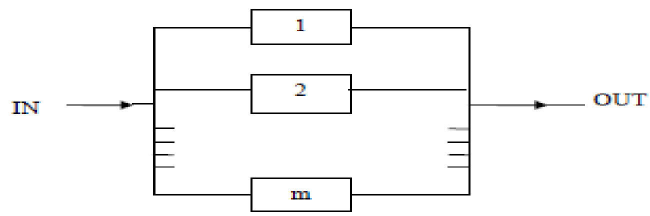

Figure 1.2: Parallel system

1.3.4.3 k out of m systems

The system will work if at least k components work out of m identical components. For systems with identical and statistically independent components, the Binomial distribution can be used to evaluate the reliability. If p is the probability of success of each component, then the probability that exactly x out of m components are successful is given by

$$p(m,x) = ^m C_x p^x (1-p)^{m-x}.$$

Hence the reliability of the system is given by

$$\sum_{i=k}^{m} \begin{pmatrix} n \\ i \end{pmatrix} p^x (1-p)^{m-x}.$$

1.3.5 Reliability and life testing

The reliability of components and systems can be predicted by various methods and procedures. Many different models and data sources are available for this purpose. However, all these models are developed based on some assumptions and approximations to simplify the procedures. Therefore, the predicted reliabilities can be considered good only as an initial approximate estimate of the actual performance of the component or system for comparative studies. The actual performance (reliability) of the system or component can be evaluated from the field data. This is the best approach for reliability evaluation. However, this is easier to say than to do in practice. Field data collection, storage, and retrieval need tremendous efforts. Effective and honest cooperation of industry, service stations, and costumers is needed for this to happen. In many cases, actual field data are not kept in a proper format. This results in data grouping and associated loss of vital information. Under these situations, the most practical and acceptable alternative is data generation in laboratories by conducting life tests.

The most important objective of life testing is to generate failure data. There are many more objectives for conducting life tests in the laboratory. It depends on the purpose of conducting the life test. Life testing may be for engineering failure analysis, cause analysis, identification of failure mechanisms, weak links, and other specific purpose of interest.

1.3.6 Some life testing methods

Some of the important life testing methods are briefly discussed below:

1.3.6.1 Life test with censoring

The procedure adopted for terminating a life test is called censoring. Censoring of data arises when exact lifetimes are known only for a portion of the individuals under study. A life test will get terminated when all units undergoing a test fail. However, it might take a lot of time and immense effort for this to happen in practice, and sometimes it is even not possible to attain failures of all units under the specified test. Therefore, censoring methods are developed for terminating life tests, and thereby to study the lifetime characteristics of units in the population.

Observations will need to be censored if they are above certain pre-defined bounds. Different types of censoring schemes are available in the literature. Some of the methods used for censoring in reliability test plans are discussed below:
Type-I censoring: In Type-I censoring, components (units) are tested for a predetermined duration of time. When a component fails before this fixed time it is replaced with an identical component and testing is continued. The total number of failures during this test period is noted. Let $T_1,\ T_2, \cdots,\ T_n$ be independent, identically distributed random variables each with cumulative distribution function F. Assume also that t_c is a fixed censoring time. Instead of observing $T_1,\ T_2, \cdots,\ T_n$, observe

$Y_1,\ Y_2, \cdots,\ Y_n$, where

$$Y_i = \begin{cases} T_i, & if\ T_i \leq t_c \\ t_c, & if\ T_i > t_c \end{cases} .$$

Type-II censoring: In this procedure, the number of failures is predetermined, and components (units) are tested until the required number of failures is achieved. The time duration between each failure is observed and noted. Let $r < n$ be fixed, and let $T_{(1)},\ T_{(2)}, \cdots,\ T_{(n)}$ be the order statistics of $T_1,\ T_2, \cdots,\ T_n$. Observation ceases after $r - th$ failure, so one can only observe $T_{(1)},\ T_{(2)}, \cdots,\ T_{(n)}$. The full ordered observed sample can be written as: $Y_1 = T_{(1)},\ Y_2 = T_{(2)}, \cdots, Y_r = T_{(r)},\ \cdots,\ Y_{r+1} = T_{(r)}\ \cdots,\ Y_n = T_{(r)}$.

Mixed censoring: In mixed censoring, a predetermined number of units of a component are tested for a predetermined unit time. If a component survives the test for this unit time, then the component is classified as a successful unit; if a component fails the test during this unit time, then it is declared as a failed unit. In either case, a new component is immediately put on the test, and this is continued until the predetermined number of components is tested.

Progressive Type-II censoring: In life testing experiments, data is often censored. Among the different censoring schemes, Type-I and Type-II censorings are the most popular. Unfortunately, in these censoring schemes, it is not possible to withdraw live units during the experiment. A generalization of the classical Type-II censoring scheme allows for the withdrawal of live units during the experiment, and it is called the progressive Type-II censoring scheme. In this scheme, n units are placed on a life test for observing a predetermined m number of failures. At the time of the first failure, R_1 of the remaining $n - 1$ surviving units are randomly removed from the experiment. At the time of the second failure, R_2 of the remaining $n-2-R_1$ units are randomly removed from the experiment. Finally, at the time of the $m-th$ failure, all the remaining units $R_m = n-m-R_1-\cdots-R_{m-1}$ are removed from the experiment.

1.3.6.2 Life test with replacement

In tests with replacement, life testing is started with n units. Whenever a unit fails, it is replaced with a new unit and the testing is continued without any halt. The test is terminated either after the pre-decided time or after the occurrence of the r-th failure. Test with replacement will generate more data over a period of time as compared to the data generated by tests without replacement over the same time period.

1.3.6.3 Simulated life tests

In this procedure, units are tested in simulated operating conditions; equipment such as test chambers are used for obtaining the simulated operating conditions. Some of the factors that significantly influence the operating conditions are temperature, humidity, pressure, etc. The actual tests are carried out with various combinations of these factors. Best results are achieved through the proper control of simulated test conditions.

1.3.6.4 Accelerated life testing

Most of the highly reliable products manufactured in the industry do not fail very easily. To obtain failure data of such products further techniques are required. Accelerated life testing is one such technique to obtain faster failures. A variety of procedures is used to accelerate failures due to the difficulties encountered in performing life tests with time deadlines. Accelerated tests are designed and carried out for accelerating failures so that more number of units fail in a short test duration. Compressed-time test and advanced stress tests are two different types of accelerated life tests. In compressed-time testing, loads and environmental stresses on the product are maintained at the same level as in normal use but the product is used more frequently in the life test than it would normally be used. In advanced stress test, life testing is conducted at higher stress levels to obtain more data within a short time. Some of the common accelerated tests are design qualification test, environmental stress screening, etc.

1.3.7 Methods of estimation of parameters

1.3.7.1 Maximum likelihood estimation

A statistical model is specified by identifying an appropriate sampling distribution. The probability of observing an outcome of a future experiment conducted on a sample of items drawn from a population of interest is known as the sampling distribution. Once the experiment is conducted, the sampling distribution is a function of the unknown parameters and is called the likelihood function. This function contains all information of the data that is relevant for estimating the unknown model parameters. These model parameters are estimated by finding the value of the parameters that maximize the value of the likelihood function. Such estimates are called maximum likelihood estimates. These estimates make the observed data as likely as possible.

For computational convenience, the logarithm of the likelihood function is maximized instead of the likelihood function itself. This is performed because the same value maximizes both the functions. The logarithm of the likelihood function is called as a log-likelihood function. When the observations are conditionally independent, the log-likelihood function is the sum of the logarithm of the density values evaluated at each observation. Taking the derivative of the log-likelihood function with respect to the unknown parameter and equating it to zero gives the maximum likelihood estimator of the parameter. the MLE has always been a good choice for an estimator due to the following reasons, namely, (i) ease of calculation, (ii) consistent estimators, (iii) functions of sufficient statistics, (iv) asymptotic normality under regulated conditions.

1.3.7.2 Bayesian estimation of reliability

In Bayesian reliability analysis, the statistical model consists of two parts, namely likelihood function, and the prior distribution. The function constructed from the sampling distribution of the data, defined using the probability density function assumed by the data is called the likelihood function. It is a function of the unknown parameters. Now in the Bayesian analysis, the parameters in the likelihood function are treated as unknown quantities. The uncertainty in these parameters is described using probability density functions. Before the data is being analyzed, the distribution that represents our knowledge about these parameters is known as prior distribution. Thus the likelihood function and the prior distribution are the basis for parameter estimation and inference. The posterior distribution calculated using the Bayes theorem uses the likelihood function and the prior distribution to describe the uncertainty associated with the parameter. The posterior distributions are true probability statements about the unknown parameters. Thus the likelihood function, prior distribution, Bayes theorem and posterior distribution makes Bayesian reliability analysis easy to describe and derive estimates that are easy to interpret and use.

The above mentioned basic definitions and some methods discussed across various subsections are as mentioned in [73], [75], [18], [28], [71], [26], [65], [92] and [41].

1.4 Thesis summary

This thesis deals with the design of reliability test plans and estimation using classical and Bayesian methodology. The thesis is organized into seven chapters, including introduction as the first chapter and concluding remarks and scope for future work as the last chapter. A summary of the remaining chapters are presented below:

In Chapter 2, optimal reliability test plans are designed for a highly reliable parallel system with n different components. The data are obtained from Type-II censoring, with the assumption that components have lifetimes that are exponentially distributed random variables, and failure rates of components depend on k covariates such as room temperature, humidity, and pressure. An unbiased estimator and maximum likelihood estimators are used to obtain system reliability. Using both unbiased estimator and maximum likelihood estimator, optimal reliability test plans are designed satisfying the probability requirements of Type-I and Type-II error constraints. Several numerical results are illustrated and compared with the existing results. The proposed test plan has significant savings in testing costs as compared to that in the existing plans in literature.

Chapter 3 presents optimal reliability test plans for a series system with n different components. The data are obtained from Type-II censoring, with the assumption that components have lifetimes that are exponentially distributed random variables, and failure rates of components depend on k covariates. Initially, a reliability test plan is constructed using the system reliability estimate obtained from unbiased estimator of failure rate. Through this work, it is shown that for a series system, the optimum design depends on the cost of individual components and that all components in the system need not be tested equally. Unlike most of the plans available in the literature, in the proposed plan, the acceptance constant d^* and the optimum sample size for each component depends upon the testing costs of individual components. The proposed test plan has an advantage of about 79% savings in testing costs as compared to that in the existing literature. Secondly, a maximum likelihood estimators of failure rates are used to obtain MLE of system reliability. The maximum-total-expected-testing-cost expression is obtained. Since it is difficult to minimize random testing cost involved in Type-II censoring, an optimization problem is formulated to minimize maximum-total-expected-testing-cost, and optimal parameters are obtained subject to the requirements of Type-I and Type-II error constraints. It is observed that optimum design depends upon the cost of individual component and the number of components in the system. The acceptance constant d_M^* and the optimum sample size for each component are depending upon testing costs. A simulation study conducted shows that in the proposed model, the derived sampling plans meet the specified risks α and β. Also, from sensitivity analysis studies, it is clear that the model is sensitive to its parameters and the corresponding output is stable. It is observed that the percentage of components to be tested for failure is reduced by about 96%. Also, there is a significant reduction in testing costs of about 77% as compared to that with Rajgopal and Mazumdar (1996: [82]). Similarly, it is observed that there is a significant reduction in testing costs of about 96% as compared to that with Vellaisamy and Kumar (2010: [73]).

Chapter 4 discusses the design of optimal reliability test plan for a series and parallel system with failure rates of components as random variables, having Quasi-density. The data are obtained through the Type-I censoring scheme, and the reliability estimator is obtained by estimating a Bayesian estimator of component failure rates. Some numerical examples are also computed to illustrate the Bayesian approach of estimating system reliability and thereby to test the system reliability. The proposed Bayesian plan has about 70% savings in total testing costs.

Chapter 5 focuses on constructing acceptance sampling plan for Weibull distribution using constant stress partially accelerated life test by obtaining data using Type-II censoring scheme. Two different stress relations, namely, linear and Arrhenius relationships are considered. The maximum likelihood estimates of an unknown parameter of Weibull distribution and acceleration factor are obtained in a linear life-stress model, and corresponding optimal acceptance sampling plan is developed. Similarly, the maximum likelihood estimates of model parameters are obtained in case of Arrhenius life-stress relationship, and the corresponding optimal sampling plan is developed satisfying the requirements of Type-I and Type-II error constraints. Since the test cost involved in constructing an acceptance sampling plan is random, an expression for expected testing cost is given. It is observed that when the values of producer's and consumer's risks decrease, the testing cost is increasing in both the linear and Arrhenius life-stress relationships, and it is noted that the total expected testing cost is less in Arrhenius model as compared to linear model. It is also observed that Arrhenius model is more cost-effective than the linear model.

Chapter 6 discusses the general degradation path model, the exponential degradation model, and illustrates the reliability estimation methodology. Bayes estimate of scale parameter α of Weibull distribution of degradation parameter θ (rate of degradation) is obtained. Bayesian reliability of first kind and second kind for the system are computed under informative and noninformative priors. The bootstrap method is used for finding standard error of Bayes estimator of α with respect to both informative and noninformative priors. Gibbs sampling procedure is applied for estimating reliability. In both informative and non-informative prior cases, it is observed that the estimated system reliability approaches the actual reliability when the sample size increases. In the case of Quasi prior, if $k = 1$, we will get maximum likelihood estimator of system reliability. It is also observed through a numerical computation that, the estimated reliability is almost the same as the original reliability when $k = 1$. Several numerical examples are presented, and it is observed that in both informative and noninformative prior cases, the standard error decreases when the bootstrap sample size increases. Thus one can have a Bayesian reliability estimator by using minimum variance Bayes estimator of α.

Chapter 7 presents conclusions of research work and scope for future research.

Chapter 2

Design of Optimal Reliability Test Plan for a Parallel System in the Presence of Covariates

2.1 Introduction

In most real-life situations, the lifetime of a system (series/parallel/mixed type) is influenced by risk factors, which are called covariates. The fundamental notion of reliability theory is the failure time of a system and its covariates. These covariates change stochastically and may influence and/or indicate the failure time. For example, modeling crack size of the fuselage of an aircraft that is affected by continuously varying covariates such as temperature, pressure, etc. is necessary for assessing reliability. Hence, it is difficult to assess reliability with traditional life tests that record only failure time. Several of the component reliability test plans addressed in the literature so far, are such that, either optimal test times for all the components are determined (for

[*]Some results of this chapter are published in the following papers.

P. N. Bajeel and M. Kumar.:*A component reliability test plan for a parallel system with failure rate as the exponential function of covariates.* Mathematics in Engineering, Science & Aerospace. Cambridge Scientific Publishers, Cambridge; I&S Publishers, USA. Vol. 6 (2), pp. 177–191, (2015) - (Scopus indexed).

P. N. Bajeel and M. Kumar.:*Optimal Reliability Test Plan for a Parallel System with Covariate Information.* Proceedings of National Conference on Emerging Trends in Statistical Research: Issues and Challenges held at Department of Statistics, Pondicherry University, Feb. 16-17, Narosa Publishing House Pvt. Ltd., ISBN: 978-81-8487-558-4. pp. 61-71, (2016).

series/parallel systems) by assuming components lifetimes are independently and exponentially distributed random variables, or optimum number of each component to be tested is determined (for series/parallel systems) by assuming that their lifetimes to be exponentially distributed random variables with constant unknown failure rates. Some of the references related to this topic are [9,11,14,29,36,59,82,83,99]. The optimal test times or the optimal number of components to be tested, are reported under various situations in these papers are not realistic. In other words, they are supposed to be used under normal working conditions regardless of the environment in which component testing is carried out. Consider the reliability test plan proposed in [10], in which failure rate changes dynamically with respect to time. Here, components are assumed to have piecewise constant failure rates.

The test programs in general, can be conducted at various levels; for example, component, subsystem, and system level. The advantages of component-level tests are described in [80,81]. In short, system level tests are generally more expensive and time-consuming. Thus most often, component testing provides a superior alternative, as long as the system-level objectives are followed. More discussions on component reliability test plans can be seen in [29].

In the paper [79], an optimization problem is formulated for deriving the minimum-cost component-testing plans when a Type-II censored component-test procedure is used for a parallel system. Rajgopal and Mazumdar (1996: [82]) designed a reliability test plan under Type-II censoring scheme. In their plan, each component has to be tested equally, and the optimal sample size does not depend upon the number of components in the system and, hence it is enough to test a single component system. Yan and Mazumdar (1987: [100]) designed a reliability test plan for a parallel system with Type-II censoring. They also have considered the failure rate as constant, but unknown. They provide a criterion for accepting or rejecting the system based on the product of the total times on the test for each component. Rajgopal and Mazumdar (1988: [79]) considered the problem of acceptance testing for a parallel system of different components having constant failure rates. They use both Type-I and Type-II censoring scheme for obtaining data. Their paper provides a criterion for accepting or rejecting the system based on the sum of the logarithms of the total times on test for each component.

This chapter explores the reliability test plans for a highly reliable parallel system consisting of n different components with the assumption that components have lifetime that are exponentially distributed random variables, and the failure rates of the components depend on k covariates $x_1, x_2, ..., x_k$, such as room temperature, humidity, pressure, etc. through exponential relationship. The data is obtained through Type-II censoring. The failure rate $\lambda_i(x)$ can be expressed as

$$\lambda_i(x) = e^{\sum\limits_{j=1}^{k} \beta_{ij} x_j} = e^{\beta_{i1}x_1 + \beta_{i2}x_2 + ... + \beta_{ik}x_k},$$

where $\beta_{ij} \in \mathbb{R}$, $i = 1, 2, ..., n; j = 1, 2, ..., k$ and $x = (x_1, x_2, ..., x_k)$. Then the parallel system reliability for unit time is given by $R = 1 - \prod\limits_{i=1}^{n}(1 - e^{-\lambda_i(x)})$. Since the system is highly reliable, the system reliability can be approximated as

$$R = 1 - \prod_{i=1}^{n}(1 - e^{-\lambda_i(x)}) \simeq 1 - \prod_{i=1}^{n} \lambda_i(x).$$

This chapter is organized as follows: In Section 2.2, using unbiased estimator of failure rates, a component reliability test plan for a parallel system with failure rate as the exponential function of covariates is designed. Definition of acceptance rule and normal approximation of the distribution of test statistics using Delta method are given in Sections 2.2.1 and 2.2.2 respectively. The problem in this case, is formulated in Section 2.2.3 and the corresponding optimal design is developed. An algorithm is given in Section 2.2.4 to obtain the optimal design. Numerical results and comparisons are given in Section 2.2.5. In Section 2.3, another optimal reliability test plan for a parallel system with covariate information is constructed, using MLEs of failure rates. The maximum likelihood estimator of failure rates are obtained in Section 2.3.2. In Section 2.3.3, an acceptance rule based on maximum likelihood estimator of system reliability is defined. Normal approximation for distribution of test statistics obtained in Section 2.3.3, is explained in Section 2.3.4. A concept of acceptable reliability interval (ARI) and unacceptable reliability interval (URI) are defined in Section 2.3.1. The optimization problem is formulated and optimal design is developed in Section

2.3.5. In Section 2.3.6, an algorithm is given to illustrate the optimal design developed in Section 2.3.5. Numerical illustration and comparative studies are discussed in Section 2.3.7. The conclusions are drawn in Section 2.4. The statements and proofs of Lemma's are included in the appendix.

2.2 Component reliability test plan with failure rate as the exponential function of covariates: Unbiased estimation approach.

In this section, reliability test plans for a highly reliable parallel system consisting of n different components with the assumption that components have lifetime that are exponentially distributed random variables, and the failure rates of the components depend on k covariates $x_1, x_2, ..., x_k$, such as room temperature, humidity, pressure (see, [88]) through exponential relationship is designed. The failure rate $\lambda_i(x)$ can be expressed as

$$\lambda_i(x) = e^{\sum\limits_{j=1}^{k} \beta_{ij} x_j} = e^{\beta_{i1}x_1 + \beta_{i2}x_2 + ... + \beta_{ik}x_k},$$

where $\beta_{ij} \in \mathbb{R}$, $i = 1, 2, ..., n; j = 1, 2, ..., k$ and $x = (x_1, x_2, ..., x_k)$. Then the parallel system reliability for unit time is given by $R = 1 - \prod\limits_{i=1}^{n}(1 - e^{-\lambda_i(x)})$. Since the system is highly reliable, the system reliability can be approximated as

$$R = 1 - \prod_{i=1}^{n}(1 - e^{-\lambda_i(x)}) \simeq 1 - \prod_{i=1}^{n} \lambda_i(x).$$

The analysis and construction of reliability test plan is based on unbiased estimators of failure rates. The aim is to obtain system reliability estimate based on these unbiased estimators.

A system is said to be satisfactory for unit time period, if R, the survival probability is greater than or equal to R_1, the acceptable reliability level (ARL), and it is said to be unsatisfactory if R is less than or equal to R_0, the unacceptable reliability level (URL), where R_0 and R_1 are constants such that $0 < R_0 < R_1 < 1$. Then the following relations are true.

$$R \geq R_1 \Rightarrow \prod_{i=1}^{n} e^{\sum\limits_{j=1}^{k} \beta_{ij} x_j} \leq (1 - R_1) \Rightarrow \sum_{i=1}^{n} \sum_{j=1}^{k} \beta_{ij} x_j \leq ln(1 - R_1).$$

$$R \leq R_0 \Rightarrow \prod_{i=1}^{n} e^{\sum\limits_{j=1}^{k} \beta_{ij} x_j} \geq (1 - R_0) \Rightarrow \sum_{i=1}^{n} \sum_{j=1}^{k} \beta_{ij} x_j \geq ln(1 - R_0).$$

Let X_i denote the lifetime of $i - th$ component, $1 \leq i \leq n$, which is independent and follows Exponential distribution with parameter $\lambda_i(x)$. Assume that the prior information in the form of upper bound $u_i(x)$ for each failure rate $\lambda_i(x)$ is known from the previous experiments, and which depends on covariates. Test the $i-th$ component r_i times for failure. Then the sum of r_i failure times is denoted by $T_i = \sum\limits_{j=1}^{r_i} X_{ij}$, where X_{ij} denote the lifetime of $j - th$ component of type i. Note that T_i follows Gamma distribution with parameters $\lambda_i(x)$ and r_i, $1 \leq i \leq n$. Then the probability density function of T_i is given by

$$f(t_i,\ \lambda_i(x),\ r_i) = \frac{(\lambda_i(x))^{r_i}}{\Gamma r_i} e^{-\lambda_i(x) t_i} t_i^{r_i - 1}.$$

In this section, λ_i is considered as a function of covariates and the unbiased estimator for $\lambda_i(x)$ is given by $\hat{\lambda}_i = \frac{r_i - 1}{T_i}$. Note that the same estimator was obtained in [82] when λ_i's are constant.

2.2.1 An acceptance rule based on $\hat{\lambda}_i$

Since for a highly reliable parallel system, the reliability of the system for unit time period is approximately given by $R \simeq 1 - \prod\limits_{i=1}^{n} \lambda_i(x)$. Therefore, accept the system iff the estimator of the system reliability $\hat{R} = 1 - \prod\limits_{i=1}^{n} \dfrac{r_i - 1}{T_i}$ grater than or equal to some number d, where $d \in (0, 1)$. Then

$$\hat{R} \geq d \Rightarrow 1 - \prod_{i=1}^{n} \frac{r_i - 1}{T_i} \geq d \Rightarrow \sum_{i=1}^{n} ln\left(\frac{r_i - 1}{T_i}\right) \leq ln(1 - d).$$

That is, accept the system if and only if $S = \sum_{i=1}^{n} ln\left(\frac{r_i - 1}{T_i}\right) \leq d_0$, where $d_0 = ln(1-d)$, otherwise reject the system.

2.2.2 Normal approximation of distribution of S using Delta method

In probability and statistics, the delta method is a result concerning the approximate probability distribution for a function of an asymptotically normal statistical estimator. It was first described in 1938 by Robert Dorfman (see, [51]). Recall that T_i follows Gamma distribution with mean $\mu_i = \frac{r_i}{\lambda_i(x)}$ and variance $\sigma_i^2 = \frac{r_i}{(\lambda_i(x))^2}$. Define $g(T_i) = ln\left(\frac{r_i - 1}{T_i}\right)$, then

$$g'(T_i) = \frac{-1}{T_i},\ g(\mu_i) = \ln\left(\frac{r_i - 1}{\mu_i}\right) = \ln\left(\frac{(r_i - 1)\lambda_i(x)}{r_i}\right)$$

and $g'(\mu_i) = \frac{-\lambda_i(x)}{r_i}$. Then by Delta method $g(T_i)$ follows Normal distribution with mean $g(\mu_i) = \ln\left(\frac{r_i-1}{r_i}\lambda_i(x)\right)$ and variance

$$\left(g'(\mu)\right)^2 \sigma^2(T_i) = \left(\frac{-\lambda_i(x)}{r_i}\right)^2 \left(\frac{r_i}{(\lambda_i(x))^2}\right) = \frac{1}{r_i}.$$

That is,

$$g(T_i) \approx N\left(ln\left(\frac{r_i - 1}{r_i}\lambda_i(x)\right), \frac{1}{r_i}\right).$$

Define

$$S = g(T) = \sum_{i=1}^{n} g(T_i) = \sum_{i=1}^{n} ln\left(\frac{r_i - 1}{T_i}\right),$$

then by Lindeberg central limit theorem, S follows Normal distribution with mean $\sum_{i=1}^{n} ln\left(\frac{r_i - 1}{r_i}\lambda_i(x)\right)$ and variance $\sum_{i=1}^{n} \frac{1}{r_i}$.

2.2.3 Problem formulation and optimal design

Let the cost of testing the $i-th$ component be denoted by c_i. Note that here c_i is a constant cost, which can be fixed based on experience. Then based on Type-II censoring scheme, the total cost for testing is $C = \sum_{i=1}^{n} c_i r_i$. Similar cost expression can be seen in [82]. Then the problem is to minimize C subjected to the requirements of producer's risk and consumer's risk.

That is, the Problem P:

$$\min_{r_i} C = \sum_{i=1}^{n} c_i r_i$$

such that

$$P(\textit{Reject the system} \mid \textit{System is good}) < \alpha, \tag{2.2.1}$$

$$P(\textit{Accept the system} \mid \textit{System is bad}) \leq \beta, \tag{2.2.2}$$

where $0 < \alpha, \beta < 1$. In above Inequalities, α is usually known as producer's risk, and β is known as consumer's risk. Using the acceptance rule defined in Section 2.2.1, the constraints 2.2.1 and 2.2.2 can be written as

$$P\left(\sum_{i=1}^{n} ln\left(\frac{r_i - 1}{T_i}\right) \leq d_0 \mid \sum_{i=1}^{n}\sum_{j=1}^{k} \beta_{ij}x_j \leq ln(1 - R_1)\right) \geq 1 - \alpha, \tag{2.2.3}$$

$$P\left(\sum_{i=1}^{n} ln\left(\frac{r_i - 1}{T_i}\right) \leq d_0 \mid \sum_{i=1}^{n}\sum_{j=1}^{k} \beta_{ij}x_j \geq ln(1 - R_0)\right) \leq \beta, \tag{2.2.4}$$

Note that $\lambda_i(x) \leq u_i(x)$ for all $i = 1, 2, ...n \Rightarrow \sum_{j=1}^{k} \beta_{ij}x_j \leq ln\ u_i(x)$ for all $i = 1, 2, ...n$. Then in terms of probability of acceptance, constraint 2.2.3 states that the probability of acceptance should be at least $1 - \alpha$ for all combinations of $\lambda_i(x)$ values that satisfy $\sum_{i=1}^{n}\sum_{j=1}^{k} \beta_{ij}x_j \leq ln(1 - R_1)$. That is, the minimum probability of acceptance over all such $\lambda_i(x)$ should exceed $1 - \alpha$. The constraint 2.2.4 states that the probability of acceptance should be at most β for all combinations of $\lambda_i(x)$ values that satisfy $\sum_{i=1}^{n}\sum_{j=1}^{k} \beta_{ij}x_j \geq ln(1 - R_0)$. That is, the maximum probability of acceptance over all such $\lambda_i(x)$ should not exceed β. Also $\sum_{j=1}^{k} \beta_{ij}x_j \leq ln\ u_i(x)$ for all $i = 1, 2, ...n$. Therefore constraints 2.2.3 and 2.2.4 can be rewritten as

$$\min_{\beta_{ij}} P\left(\sum_{i=1}^{n} ln\left(\frac{r_i - 1}{T_i}\right) \leq d_0 \mid \sum_{i=1}^{n}\sum_{j=1}^{k} \beta_{ij}x_j \leq ln(1 - R_1)\right) \geq 1 - \alpha, \tag{2.2.5}$$

$$\max_{\beta_{ij}} P\left(\sum_{i=1}^{n} ln\left(\frac{r_i-1}{T_i}\right) \leq d_0 \mid \sum_{i=1}^{n}\sum_{j=1}^{k}\beta_{ij}x_j \geq ln(1-R_0)\right) \leq \beta. \tag{2.2.6}$$

with the common upper bound condition $\sum_{j=1}^{k}\beta_{ij}x_j \leq ln\ u_i(x)$ for all $i = 1, 2, ..., n$. Recall that $\sum_{i=1}^{n} ln\left(\frac{r_i-1}{T_i}\right)$ has mean $\sum_{i=1}^{n} ln\left(\frac{r_i-1}{r_i}\lambda_i(x)\right)$ and variance $\sum_{i=1}^{n}\frac{1}{r_i}$. Let $Z = \frac{\sum_{i=1}^{n} ln\left(\frac{r_i-1}{T_i}\right) - \sum_{i=1}^{n} \ln\left(\frac{r_i-1}{r_i}\right) - \sum_{i=1}^{n}\sum_{j=1}^{k}\beta_{ij}x_j}{\sqrt{\sum_{i=1}^{n}\frac{1}{r_i}}} \sim N(0,\ 1)$. Then using normality, constraints 2.2.5 and 2.2.6 can be written as

$$\min_{\beta_{ij}} P\left(Z \leq \frac{d_0 - \sum_{i=1}^{n}\ln\left(\frac{r_i-1}{r_i}\right) - \sum_{i=1}^{n}\sum_{j=1}^{k}\beta_{ij}x_j}{\sqrt{\sum_{i=1}^{n}\frac{1}{r_i}}} \mid \sum_{i=1}^{n}\sum_{j=1}^{k}\beta_{ij}x_j \leq ln(1-R_1)\right) \geq 1-\alpha, \tag{2.2.7}$$

$$\max_{\beta_{ij}} P\left(Z \leq \frac{d_0 - \sum_{i=1}^{n}\ln\left(\frac{r_i-1}{r_i}\right) - \sum_{i=1}^{n}\sum_{j=1}^{k}\beta_{ij}x_j}{\sqrt{\sum_{i=1}^{n}\frac{1}{r_i}}} \mid \sum_{i=1}^{n}\sum_{j=1}^{k}\beta_{ij}x_j \geq ln(1-R_0)\right) \leq \beta \tag{2.2.8}$$

with the common upper bound condition $\sum_{j=1}^{k}\beta_{ij}x_j \leq ln\ u_i(x)$ for all $i = 1, 2, ..., n$. Since the cumulative distribution function of standard normal random variable is strictly increasing function in its parameters, the constraints 2.2.7 and 2.2.8 can be written as

$$\min_{\beta_{ij}}\left(\frac{d_0 - \sum_{i=1}^{n}\ln\left(\frac{r_i-1}{r_i}\right) - \sum_{i=1}^{n}\sum_{j=1}^{k}\beta_{ij}x_j}{\sqrt{\sum_{i=1}^{n}\frac{1}{r_i}}} \mid \sum_{i=1}^{n}\sum_{j=1}^{k}\beta_{ij}x_j \leq ln(1-R_1)\right) \geq Z_{1-\alpha}, \tag{2.2.9}$$

$$\max_{\beta_{ij}}\left(\frac{d_0 - \sum_{i=1}^{n}\ln\left(\frac{r_i-1}{r_i}\right) - \sum_{i=1}^{n}\sum_{j=1}^{k}\beta_{ij}x_j}{\sqrt{\sum_{i=1}^{n}\frac{1}{r_i}}} \mid \sum_{i=1}^{n}\sum_{j=1}^{k}\beta_{ij}x_j \geq ln(1-R_0)\right) \leq Z_{\beta}, \tag{2.2.10}$$

where, $d_0 \in (\ln(1-R_1),\ \ln(1-R_0))$ with the common upper bound condition

$$\sum_{j=1}^{k} \beta_{ij} x_j \leq ln\ u_i(x),\ i = 1, 2, ..., n.$$

Now consider the left hand side of the constraint 2.2.9

$$\min_{\beta_{ij}} \frac{d_0 - \sum\limits_{i=1}^{n} \ln\left(\frac{r_i - 1}{r_i}\right) - \sum\limits_{i=1}^{n}\sum\limits_{j=1}^{k} \beta_{ij} x_j}{\sqrt{\sum\limits_{i=1}^{n} \dfrac{1}{r_i}}} \tag{2.2.11}$$

such that

$$\sum_{i=1}^{n}\sum_{j=1}^{k} \beta_{ij} x_j \leq ln(1 - R_1), \tag{2.2.12}$$

$$\sum_{j=1}^{k} \beta_{ij} x_j \leq ln\ u_i(x)\ \forall\ i = 1, 2, ..., n. \tag{2.2.13}$$

Now consider the two different cases presented below:

Case 1: If $\sum\limits_{i=1}^{n} \ln u_i(x) \geq \ln(1 - R_1)$.

Under this case, the constraint 2.2.12 implies that it is enough to minimize the objective function 2.2.11 subjected to $\sum\limits_{j=1}^{k} \beta_{ij} x_j \leq ln\ u_i(x)$ for all $i = 1, 2, ..., n$. Since this is a convex linear programming problem in β_{ij}, the optimum lies on the boundary and the minimum value of the objective function with the two constraints is given by

$$\frac{d_0 - \sum\limits_{i=1}^{n} \ln\left(\frac{r_i - 1}{r_i}\right) - \sum\limits_{i=1}^{n} \ln u_i(x)}{\sqrt{\sum\limits_{i=1}^{n} \dfrac{1}{r_i}}}$$

Then the constraint 2.2.9 can be written as

$$\frac{d_0 - \sum\limits_{i=1}^{n} \ln\left(\frac{r_i - 1}{r_i}\right) - \sum\limits_{i=1}^{n} \ln u_i(x)}{\sqrt{\sum\limits_{i=1}^{n} \dfrac{1}{r_i}}} \geq Z_{1-\alpha}. \tag{2.2.14}$$

Since α is a small value less than 0.5, $Z_{1-\alpha} > 0$. Therefore above inequality 2.2.14 can be written as

$$(Z_{1-\alpha})^2 \sum_{i=1}^{n} \frac{1}{r_i} \leq \left(d_0 - \sum_{i=1}^{n} \ln\left(\frac{r_i - 1}{r_i}\right) - \sum_{i=1}^{n} \ln u_i(x) \right)^2. \tag{2.2.15}$$

Case 2: If $\sum\limits_{i=1}^{n} \ln u_i(x) < \ln(1 - R_1)$.

In this case, the optimum lies on the boundary, because the objective function and the constraints are linear convex functions. Therefore, it is enough to find the minimum value of the objective function 2.2.11 subjected to the constraints

$$\sum_{i=1}^{n} \sum_{j=1}^{k} \beta_{ij} x_j = ln(1 - R_1), \tag{2.2.16}$$

and

$$\sum_{i=1}^{n} \beta_{ij} x_j = \ln u_i(x) \ \forall \ i = 1, 2, \ldots, n. \tag{2.2.17}$$

This is a linear minimization problem with equality constraints. Clearly, the optimum attains in any one of the corner points in the feasible region and the minimum value of the objective function 2.2.11 with above two constraints 2.2.16 and 2.2.17 is given by

$$\frac{d_0 - \sum\limits_{i=1}^{n} \ln\left(\frac{r_i-1}{r_i}\right) - \ln(1 - R_1)}{\sqrt{\sum\limits_{i=1}^{n} \frac{1}{r_i}}}.$$

Then constraint 2.2.9 can be written as

$$\frac{d_0 - \sum\limits_{i=1}^{n} \ln\left(\frac{r_i-1}{r_i}\right) - \ln(1 - R_1)}{\sqrt{\sum\limits_{i=1}^{n} \frac{1}{r_i}}} \geq Z_{1-\alpha}. \tag{2.2.18}$$

Since α is a small value less than 0.5, $Z_{1-\alpha} > 0$. Therefore, above inequality 2.2.18 can be written as

$$(Z_{1-\alpha})^2 \sum_{i=1}^{n} \frac{1}{r_i} \leq \left(d_0 - \sum_{i=1}^{n} \ln\left(\frac{r_i - 1}{r_i}\right) - \ln(1 - R_1) \right)^2. \tag{2.2.19}$$

Similarly, when $\beta < 0.5$, the constraint 2.2.10 can be written under Case 1 and Case 2 as,

$$(Z_\beta)^2 \sum_{i=1}^{n} \frac{1}{r_i} \le \left(d_0 - \sum_{i=1}^{n} \ln\left(\frac{r_i - 1}{r_i}\right) - \sum_{i=1}^{n} \ln u_i(x) \right)^2. \tag{2.2.20}$$

The optimization problem with constraints 2.2.9 and 2.2.10 can be written as two separate problems $\mathscr{P}_1$ and $\mathscr{P}_2$ as follows:

Problem $\mathscr{P}_1$: If Case 1 is true, then

$$\min_{r_i} C = \sum_{i=1}^{n} c_i r_i$$

such that

$$(Z_{1-\alpha})^2 \sum_{i=1}^{n} \frac{1}{r_i} \le \left(d_0 - \sum_{i=1}^{n} \ln\left(\frac{r_i - 1}{r_i}\right) - \sum_{i=1}^{n} \ln u_i(x) \right)^2. \tag{2.2.21}$$

$$(Z_\beta)^2 \sum_{i=1}^{n} \frac{1}{r_i} \le \left(d_0 - \sum_{i=1}^{n} \ln\left(\frac{r_i - 1}{r_i}\right) - \sum_{i=1}^{n} \ln u_i(x) \right)^2. \tag{2.2.22}$$

This is an integer programming problem and can easily be solved.

Problem $\mathscr{P}_2$: If Case 2 is true, then

$$\min_{r_i} C = \sum_{i=1}^{n} c_i r_i$$

such that

$$(Z_{1-\alpha})^2 \sum_{i=1}^{n} \frac{1}{r_i} \le \left(d_0 - \sum_{i=1}^{n} \ln\left(\frac{r_i - 1}{r_i}\right) - \ln(1 - R_1) \right)^2. \tag{2.2.23}$$

$$(Z_\beta)^2 \sum_{i=1}^{n} \frac{1}{r_i} \le \left(d_0 - \sum_{i=1}^{n} \ln\left(\frac{r_i - 1}{r_i}\right) - \sum_{i=1}^{n} \ln u_i(x) \right)^2. \tag{2.2.24}$$

Again, this is an integer programming problem and can easily be solved.

2.2.4 An algorithmic procedure to solve the Problem P

Let z_c^*, r_i^*, d_0^* and M denotes the optimum cost, optimum number of failures of $i-th$ component, optimum value of d_0 and a large positive value respectively. Divide $(\ln(1 - R_1),\ \ln(1 - R_0))$ into l equal subintervals of length $\delta(> 0)$.

Step 1: Read $n,\ k,\ x_j,\ c_i,\ R_0,\ R_1,\ \alpha,\ \beta,\ l$ and $u_i(x)$ for $i = 1, 2, ..., n$ and

$j = 1, 2, ..., k$.
Step 2: Set $z_c^* = M$.
Step 3: Set $\delta = \frac{\ln(1-R_0)-\ln(1-R_1)}{l}$.
Step 4: If $\sum\limits_{i=1}^{n} ln\,(u_i(x)) \geq ln(1-R_1)$, then go to Step 5. Else go to Step 17.
Step 5: Set $d_0 = \ln(1-R_1)$.
Step 6: If $d_0 \leq \ln(1-R_0)$, then go to next Step 7. Else go to Step 10.
Step 7: Solve the integer optimization problem with constraints 2.2.21 and 2.2.22.
Step 8: If $z < z_c^*$, then replace $z_c^* = z,\ r_i^* = r_i\ \forall\ i = 1, 2, \cdots, n$ and $d_0^* = d_0$.
Step 9: Set $d_0 = d_0 + \delta$ and go to Step 6.
Step 10: If $\sum\limits_{i=1}^{n} lnu_i(x) < ln(1-R_1)$, then go to next Step 11. Else go to Step 16.
Step 11: Set $d_0 = \ln(1-R_1)$.
Step 12: If $d_0 \leq \ln(1-R_0)$, then go to next Step 13. Else go to Step 16.
Step 13: Solve the integer optimization problem with constraints 2.2.23 and 2.2.24.
Step 14: If $z < z_c^*$, then replace $z_c^* = z,\ r_i^* = r_i\ \forall\ i = 1, 2, \cdots, n$ and $d_0^* = d_0$.
Step 15: Set $d_0 = d_0 + \delta$ and go to Step 12.
Step 16: Display $z_c^*,\ r_i^*\ \forall\ i = 1, 2, \cdots, n$ and d_0^*.
Step 17: Exit.

2.2.5 Numerical results and comparison

In this section, the Problem P is solved using algorithm designed in Section 2.2.4. Numerical results for two component parallel system with two covariates are presented. Let $\mathbf{X} = (x_1,\ x_2)$ denote the covariate vector, $\mathbf{c} = (c_1,\ c_2)$ denote the cost vector and $\mathbf{u} = (u_1,\ u_2)$ denote the upper bound vector, where $u_1 = \frac{x_1}{4(x_1+x_2)}$ and $u_2 = \frac{x_2}{3(x_1+x_2)}$. Then, the following Table 2.1 gives the optimum cost and corresponding sample sizes for $l = 250$.

Table 2.1: Reliability test plan $R(\alpha,\ \beta,\ R_0,\ R_1,\ \mathbf{c},\ \mathbf{X})$

	α	β	R_0	R_1	$\mathbf{c}$	$\mathbf{X}$	r_1^*	r_2^*	C
1	0.0001	0.0001	0.95	0.999	(1, 2.5)	(3, 4.5)	6	5	18.5
2	0.0001	0.0001	0.99	0.999	(1.5, 2.5)	(3, 4.5)	6	4	19
3	0.05	0.05	0.99	0.999	(1.5, 2.5)	(3, 4.5)	2	2	8
4	0.0001	0.0001	0.90	0.990	(1, 2.5)	(3, 4.5)	14	12	44

Consider Example 3 in Table 2.1. Here, a two component parallel system is considered with $\alpha = 0.05,\ \beta = 0.05$. This system having covariates $x_1 = 3$ and $x_2 = 4.5$ is accepted, if the system's estimated reliability is grater than or equal to the acceptable reliability level $R_1 = 0.999$ and it is rejected if the system's estimated reliability is less than or equal to the unacceptable reliability level $R_0 = 0.99$. The cost of testing each component are $c_1 = 1.5$ and $c_2 = 2.5$ units. For this set of data, employing proposed reliability test plan based on Type-II censoring, the optimal sample size obtained for the first component is $r_1 = 2$ and the same for the second component is $r_2 = 2$. For this r_1 and r_2, the total testing cost is found to be 8 units. Also, for the case $\alpha = \beta = 0.05,\ R_0 = 0.99$ and $R_1 = 0.999$, the plan proposed in [100] has optimal sample sizes (5, 5) and corresponding testing cost 20 units, but in our proposed plan, the optimal sample sizes are (2, 2) and the corresponding testing cost is 8 units. Therefore, there is about 60% reduction in testing costs in case of Reliability test plan $R(\alpha,\ \beta,\ R_0,\ R_1,\ \mathbf{c},\ \mathbf{X})$. The optimal sample sizes obtained by the plan proposed in [72] is (2,1) and the corresponding testing cost is 5.5. Therefore, the test plan derived using unbiased estimator of failure rate has savings in testing cost compared to that obtained in [100], but the plan obtained in [72] has savings in testing costs up to 58% compared to the developed test plan $R(\alpha,\ \beta,\ R_0,\ R_1,\ \mathbf{c},\ \mathbf{X})$. The results are generated by running Visual C++ and LINGO 11 in tandem.

2.3 Component reliability test plan with failure rate as the exponential function of covariates: Maximum likelihood estimation approach.

The reliability test plan designed in Section 2.2, considers the acceptance rule based on unbiased estimator of failure rates. The system will be accepted as long as reliability estimator $\hat{R}$ greater than some positive quantity in $(0,\ 1)$. Note that in this case of design, if R, the survival probability greater than or equal to R_1, the acceptable reliability level (ARL), the system is said to be satisfactory, and if $R \leq R_0$, the unacceptable reliability level (URL), the system is said to be unsatisfactory, where $0 < R_0 < R_1 < 1$. Note that the values of R_0 and R_1 are prefixed by reliability practitioner based on past experience. For example, [100], [82], [88] and [72] proposes reliability test plans based on ARL and URL. But, this satisfactory and unsatisfactory definition become absurd when $R_0 < R < R_1$. Particularly, if R is in the right neighborhood of R_0 or in the left neighbourhood of R_1. In practical situations, due to cost constraints, it is very difficult to reject a system when the reliability of the system is just below the acceptable reliability level R_1, and to accept a system when the reliability of the system is just above the unacceptable reliability level R_0. Therefore, in this section, $\varepsilon-$relaxed definitions for satisfactory and unsatisfactory levels for a system is proposed.

Consider a parallel system with n different components. Let X_i denote the lifetime of $i-th$ component, $1 \leq i \leq n$. Assume $X_i,\ 1 \leq i \leq n$, are independent and have Exponential distribution with parameter $\lambda_i(x)$. Further, assume that $\lambda_i(x)$ are unknown and are functions of k covariates $x_1, x_2, ..., x_k$. Define $\lambda_i(x)$ as the exponential function of x, where $x = (x_1, x_2, ..., x_k)$. In most of the situations, past data play essential role in testing the reliability of the system considered. Prior information $u_i(x)$ in form of upper bound for $\lambda_i(x),\ 1 \leq i \leq n$ is used. Test the $i-th$ component r_i times for failure. Let $X_{ij}, 1 \leq i \leq n; 1 \leq j \leq k$ denote lifetime of $j-th$ component of type i. Observe that X_{ij} follows Exponential distribution with parameter $\lambda_i(x), 1 \leq i \leq n; 1 \leq j \leq k$. Then, for $1 \leq i \leq n,\ T_i = \sum\limits_{j=1}^{r_i} X_{ij}$, the sum of the r_i independent and identical exponential random variables follow Gamma distribution with shape parameter r_i and scale parameter $\lambda_i(x)$. Note that r_i is a predetermined number to be found later, satisfying usual probability requirements.

2.3.1 Definition of satisfactory and unsatisfactory system

A system is satisfactory for unit time, if the estimated system reliability $\hat{R}_s$ is in the acceptable reliability interval ARI given by $I_A = \left\{\hat{R}_s \mid R_1 - \varepsilon < \hat{R}_s < 1\right\} = \left\{\hat{R}_s \mid R_1 - \hat{R}_s < \varepsilon\right\}$, and the system is unsatisfactory for unit time, if the estimated system reliability $\hat{R}_s$ is in the unacceptable reliability interval URI given by $I_U = \left\{\hat{R}_s \mid 0 < \hat{R}_s < R_0 + \varepsilon\right\} = \left\{\hat{R}_s \mid \hat{R}_s - R_0 < \varepsilon\right\}$, where ε is small positive predetermined quantity, which can be fixed by expert opinion. Then the following relations are obtained:

$$R_1 - \hat{R}_s \in I_A \Rightarrow \prod_{i=1}^{n} e^{\sum_{j=1}^{k} \beta_{ij} x_j} \leq (1 - R_1 + \varepsilon) \Rightarrow \sum_{i=1}^{n} \sum_{j=1}^{k} \beta_{ij} x_j \leq ln(1 - R_1 + \varepsilon).$$

$$\hat{R}_s - R_0 \in I_U \Rightarrow \prod_{i=1}^{n} e^{\sum_{j=1}^{k} \beta_{ij} x_j} \geq (1 - R_0 - \varepsilon) \Rightarrow \sum_{i=1}^{n} \sum_{j=1}^{k} \beta_{ij} x_j \geq ln(1 - R_0 - \varepsilon).$$

Note that the advantage of using ARI and URI in designing reliability test plan is that the optimum design has a potential in reducing huge rejection cost of the system, as compared to existing test plans. The same is discussed in Section 2.3.7.

2.3.2 Maximum likelihood estimator for the failure rate λ_i

Since for $1 \leq i \leq n,\ T_i = \sum_{j=1}^{r_i} X_{ij}$ is a random variable having Gamma distribution with shape parameter r_i and scale parameter $\lambda_i(x)$, the probability density function of T_i is given by

$$f(t_i, \lambda_i(x), r_i) = \frac{(\lambda_i(x))^{r_i}}{\Gamma r_i} e^{-\lambda_i(x) t_i} t_i^{r_i - 1}.$$

Note that T_i's are independently distributed gamma random variables. Then the corresponding likelihood function and log-likelihood are given by

$$L(\lambda_i(x), t_i, r_i) = L = \frac{(\lambda_i(x))^{r_i}}{(\Gamma r_i)} e^{-\lambda_i(x) t_i} (t_i)^{r_i - 1},$$

and

$$logL = r_i log(\lambda_i(x)) - log(\Gamma r_i) - \lambda_i(x)t_i + (r_i - 1)log(t_i).$$

Then equating the derivative of the log-likelihood function to zero implies

$$\frac{\partial logL}{\partial \lambda_i(x)} = 0 \Rightarrow r_i \frac{1}{\lambda_i(x)} - t_i = 0 \Rightarrow r_i \frac{1}{\lambda_i(x)} = t_i \Rightarrow \hat{\lambda}_i(x) = \frac{r_i}{t_i}.$$

$\hat{\lambda}_i(x) = \frac{r_i}{t_i}$ is a maximum likelihood estimate. Then maximum likelihood estimator of $\lambda_i(x)$ is given by $\frac{r_i}{T_i}$.

2.3.3 An acceptance rule based on $\hat{\lambda}_i$

Since a highly reliable parallel system is considered, the reliability of the system for unit time period is approximately given by $R(1) = R = 1 - \prod_{i=1}^{n} \lambda_i(x)$. Accept the system iff the MLE of the system reliability $\hat{R} = 1 - \prod_{i=1}^{n} \frac{r_i}{T_i}$ is grater than or equal to some number d_0, where $d_0 \in (0, 1)$. Then

$$\hat{R} \geq d_0 \Rightarrow 1 - \prod_{i=1}^{n} \frac{r_i}{T_i} \geq d_0 \Rightarrow T = \sum_{i=1}^{n} ln\left(\frac{r_i}{T_i}\right) \leq ln(1 - d_0).$$

That is, accept the system if and only if $\sum_{i=1}^{n} ln\left(\frac{r_i}{T_i}\right) \leq d$, where $d = ln(1 - d_0)$, otherwise, reject the system.

2.3.4 Normal approximation for distribution of T using Delta method

Note that it is very difficult to obtain the exact distribution of $T = \sum_{i=1}^{n} ln\left(\frac{r_i}{T_i}\right)$, where T_i follows Gamma distribution with mean $\mu = \frac{r_i}{\lambda_i(x)}$ and variance $\sigma^2 = \frac{r_i}{[\lambda_i(x)]^2}$. Approximating the distribution of T by Normal distribution using well-known Delta method (see, [83]), is the best choice. Note that $T_i \approx N\left(\frac{r_i}{\lambda_i(x)}, \frac{r_i}{[\lambda_i(x)]^2}\right)$. Define $g(T_i) = ln\left(\frac{r_i}{T_i}\right)$, then $g'(T_i) = \frac{-1}{T_i}$, $g(\mu) = ln\lambda_i(x)$ and $g'(\mu) = \frac{-\lambda_i(x)}{r_i}$. Using properties of the asymptotic normality and efficiency of the MLE (see, [21] and [83]),

$$\sqrt{r_i}\left(\frac{r_i}{T_i} - \lambda_i\right) \sim N(0, \lambda_i^2).$$

Then, by delta method $g(T_i)$ follows Normal distribution with mean $\mu_g = g(\mu) = ln\left(\lambda_i(x)\right)$ and variance $\sigma_g = \left(g'(\mu)\right)^2 \sigma^2(T_i) = \frac{1}{r_i}$. That is,

$$g(T_i) \approx N\left(ln\left(\lambda_i(x)\right), \frac{1}{r_i}\right).$$

By using Lindeberg central limit theorem, $T = \sum_{i=1}^{n} g(T_i) = \sum_{i=1}^{n} ln\left(\frac{r_i}{T_i}\right)$ follows Normal distribution with mean $\sum_{i=1}^{n} ln\left(\lambda_i(x)\right)$ and variance $\sum_{i=1}^{n} \frac{1}{r_i}$.

2.3.5 Problem formulation and optimal design

Let the cost of testing the $i-th$ component be denoted by c_i. Then, based on Type-II censoring scheme, the total cost for testing a parallel system having n different components, is $C = \sum_{i=1}^{n} c_i r_i$. The proposed problem is to minimize C subjected to the requirements of producer's risk and consumer's risk.
That is, Problem Q:

$$\min_{r_i} C = \sum_{i=1}^{n} c_i r_i$$

such that

$$P(Reject\ the\ system \mid System\ is\ good) < \alpha, \tag{2.3.1}$$

$$P(Accept\ the\ system \mid System\ is\ bad) \leq \beta, \tag{2.3.2}$$

where $0 < \alpha, \beta < 1$. In above inequalities, α is usually known as producer's risk, and β is known as consumer's risk. Then the minimization problem can be rewritten as,

$$\min_{r_i} C = \sum_{i=1}^{n} c_i r_i$$

such that

$$P(Accept\ the\ system \mid System\ is\ good) \geq 1-\alpha, \tag{2.3.3}$$

$$P(Accept\ the\ system \mid System\ is\ bad) \leq \beta. \tag{2.3.4}$$

Now, using the acceptance rule defined in Section 2.3.3, the constraints 2.3.3 and 2.3.4 can be written as

$$P\left(\sum_{i=1}^{n} ln\left(\frac{r_i}{T_i}\right) \leq d \mid \sum_{i=1}^{n}\sum_{j=1}^{k}\beta_{ij}x_j \leq ln(1-R_1+\epsilon)\right) \geq 1-\alpha, \tag{2.3.5}$$

$$P\left(\sum_{i=1}^{n} ln\left(\frac{r_i}{T_i}\right) \leq d \mid \sum_{i=1}^{n}\sum_{j=1}^{k}\beta_{ij}x_j \geq ln(1-R_0-\epsilon)\right) \leq \beta, \tag{2.3.6}$$

$r_i > 0$ for all $i = 1, 2, ..., n$ are integers, $d = ln(1-d_0)$, $d_0 \in (0,1)$. Note that $\lambda_i(x) \leq u_i(x)$ for all $i = 1, 2, ...n \Rightarrow \sum\limits_{j=1}^{k}\beta_{ij}x_j \leq ln\ u_i(x)$ for all $i = 1, 2, ...n$. Then in terms of probability of acceptance, constraint 2.3.5 states that the probability of acceptance should be at least $1-\alpha$ for all combinations of $\lambda_i(x)$ values that satisfy $\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{k}\beta_{ij}x_j \leq ln(1-R_1+\varepsilon)$. That is, the minimum probability of acceptance over all such $\lambda_i(x)$ should exceed $1-\alpha$. The constraint 2.3.6 states that the probability of acceptance should be at most β for all combinations of $\lambda_i(x)$ values that satisfy $\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{k}\beta_{ij}x_j \geq ln(1-R_0-\varepsilon)$. That is, the maximum probability of acceptance over all such $\lambda_i(x)$ should not exceed β. Since $\sum\limits_{j=1}^{k}\beta_{ij}x_j \leq ln\ u_i(x)$ for all $i = 1, 2, ...n$, the constraints (2.3.5) and (2.3.6) can be rewritten as

$$\min_{\beta_{ij}} P\left(\sum_{i=1}^{n} ln\left(\frac{r_i}{T_i}\right) \leq d \mid \sum_{i=1}^{n}\sum_{j=1}^{k}\beta_{ij}x_j \leq ln(1-R_1+\epsilon)\right) \geq 1-\alpha, \tag{2.3.7}$$

$$\max_{\beta_{ij}} P\left(\sum_{i=1}^{n} ln\left(\frac{r_i}{T_i}\right) \leq d \mid \sum_{i=1}^{n}\sum_{j=1}^{k}\beta_{ij}x_j \geq ln(1-R_0-\epsilon)\right) \leq \beta. \tag{2.3.8}$$

with the common upper bound condition $\sum\limits_{j=1}^{k}\beta_{ij}x_j \leq ln\ u_i(x)$ for all $i = 1, 2, ..., n$. Recall that T_i is a gamma random variable with shape parameter r_i and scale parameter $\lambda_i(x)$. Let $Z \sim N(0,1)$. Then, by using Lindeberg Central Limit Theorem, with mean and variance obtained by Delta method for the random variable $T = \sum\limits_{i=1}^{n} ln\left(\frac{r_i}{T_i}\right)$ (see

Section 2.3.4), one can rewrite the constraints 2.3.7 and 2.3.8 as

$$\min_{\beta_{ij}} P\left(Z \leq \frac{d - \sum_{i=1}^{n}\sum_{j=1}^{k}\beta_{ij}x_j}{\sqrt{\sum_{i=1}^{n}\frac{1}{r_i}}} \mid \sum_{i=1}^{n}\sum_{j=1}^{k}\beta_{ij}x_j \leq ln(1-R_1+\varepsilon)\right) \geq 1-\alpha, \quad (2.3.9)$$

$$\max_{\beta_{ij}} P\left(Z \leq \frac{d - \sum_{i=1}^{n}\sum_{j=1}^{k}\beta_{ij}x_j}{\sqrt{\sum_{i=1}^{n}\frac{1}{r_i}}} \mid \sum_{i=1}^{n}\sum_{j=1}^{k}\beta_{ij}x_j \geq ln(1-R_0-\varepsilon)\right) \leq \beta, \quad (2.3.10)$$

with the common upper bound condition $\sum_{j=1}^{k}\beta_{ij}x_j \leq ln\ u_i(x)$ for all $i = 1, 2, ..., n$ and $Z = \frac{\sum_{i=1}^{n} ln\left(\frac{r_i}{T_i}\right) - \sum_{i=1}^{n}\sum_{j=1}^{k}\beta_{ij}x_j}{\sqrt{\sum_{i=1}^{n}\frac{1}{r_i}}}$. Now, using the fact that the cumulative distribution function of standard normal random variable, Z, is strictly increasing function in its arguments, the constraints 2.3.9 and 2.3.10 can be rewritten as

$$\min_{\beta_{ij}} \left(\frac{d - \sum_{i=1}^{n}\sum_{j=1}^{k}\beta_{ij}x_j}{\sqrt{\sum_{i=1}^{n}\frac{1}{r_i}}} \mid \sum_{i=1}^{n}\sum_{j=1}^{k}\beta_{ij}x_j \leq ln(1-R_1+\epsilon), \sum_{j=1}^{k}\beta_{ij}x_j \leq ln\ u_i(x)\ \ \forall i\right) \geq Z_{1-\alpha}, \quad (2.3.11)$$

$$\max_{\beta_{ij}} \left(\frac{d - \sum_{i=1}^{n}\sum_{j=1}^{k}\beta_{ij}x_j}{\sqrt{\sum_{i=1}^{n}\frac{1}{r_i}}} \mid \sum_{i=1}^{n}\sum_{j=1}^{k}\beta_{ij}x_j \geq ln(1-R_0-\epsilon), \sum_{j=1}^{k}\beta_{ij}x_j \leq ln\ u_i(x)\ \ \forall i\right) \leq Z_{\beta}. \quad (2.3.12)$$

Now consider the left hand side of the constraint 2.3.11,

$$\min_{\beta_{ij}} \frac{d - \sum_{i=1}^{n}\sum_{j=1}^{k}\beta_{ij}x_j}{\sqrt{\sum_{i=1}^{n}\frac{1}{r_i}}}$$

such that

$$\sum_{i=1}^{n}\sum_{j=1}^{k}\beta_{ij}x_j \leq ln(1-R_1+\epsilon), \tag{2.3.13}$$

$$\sum_{j=1}^{k}\beta_{ij}x_j \leq ln\ u_i(x) \quad \forall i=1,2,...,n. \tag{2.3.14}$$

If $\sum\limits_{i=1}^{n} ln\ u_i(x) \geq ln(1-R_1+\epsilon)$, then by Lemma 2.5.1, the minimum value of the objective function with constraints 2.3.13 and 2.3.14, is given by

$$\frac{d-\sum\limits_{i=1}^{n} ln\ u_i(x)}{\sqrt{\sum\limits_{i=1}^{n}\frac{1}{r_i}}}.$$

Thus the constraint 2.3.11 can be written as

$$\frac{d-\sum\limits_{i=1}^{n} ln\ u_i(x)}{\sqrt{\sum\limits_{i=1}^{n}\frac{1}{r_i}}} \geq Z_{1-\alpha}.$$

This implies

$$\sum_{i=1}^{n}\frac{1}{r_i} \leq \left(\frac{d-\sum\limits_{i=1}^{n} ln\ u_i(x)}{Z_{1-\alpha}}\right)^2. \tag{2.3.15}$$

If $\sum\limits_{i=1}^{n} ln\ u_i(x) < ln(1-R_1+\epsilon)$, then by Lemma 2.5.2, the minimum value of the objective function with constraints 2.3.13 and 2.3.14, is given by

$$\frac{d-ln(1-R_1+\epsilon)}{\sqrt{\sum\limits_{i=1}^{n}\frac{1}{r_i}}}.$$

Therefore, the constraint 2.3.11 can be written as

$$\frac{d-ln(1-R_1+\epsilon)}{\sqrt{\sum\limits_{i=1}^{n}\frac{1}{r_i}}} \geq Z_{1-\alpha}.$$

This implies

$$\sum_{i=1}^{n} \frac{1}{r_i} \leq \left(\frac{d - ln(1 - R_1 + \epsilon)}{Z_{1-\alpha}} \right)^2. \tag{2.3.16}$$

Now consider the left hand side of constraint 2.3.12

$$\max_{\beta_{ij}} \frac{d - \sum\limits_{i=1}^{n} \sum\limits_{j=1}^{k} \beta_{ij} x_j}{\sqrt{\sum\limits_{i=1}^{n} \frac{1}{r_i}}}$$

such that

$$\sum_{i=1}^{n} \sum_{j=1}^{k} \beta_{ij} x_j \geq ln(1 - R_0 - \epsilon),$$
$$\sum_{j=1}^{k} \beta_{ij} x_j \leq ln\ u_i(x) \quad \forall i = 1, 2, ..., n.$$

Then by Lemma 2.5.2, the maximum value of the objective function with the given constraints, is given by

$$\frac{d - \sum\limits_{i=1}^{n} ln\ u_i(x)}{\sqrt{\sum\limits_{i=1}^{n} \frac{1}{r_i}}}.$$

Then the constraint 2.3.12 can be rewritten as

$$\frac{d - \sum\limits_{i=1}^{n} ln\ u_i(x)}{\sqrt{\sum\limits_{i=1}^{n} \frac{1}{r_i}}} \leq Z_\beta.$$

This implies

$$\sum_{i=1}^{n} \frac{1}{r_i} \leq \left(\frac{d - \sum\limits_{i=1}^{n} ln\ u_i(x)}{Z_\beta} \right)^2. \tag{2.3.17}$$

Now, the optimization problem with constraints 2.3.11 and 2.3.12 can be written as follows with two different cases:

Case 1: If $\sum\limits_{i=1}^{n} ln\ u_i(x) \geq ln(1 - R_1 + \epsilon)$. Then the optimization problem becomes:

$$\min_{r_i} C = \sum_{i=1}^{n} c_i r_i$$

such that

$$\sum_{i=1}^{n} \frac{1}{r_i} \leq \left(\frac{d - \sum\limits_{i=1}^{n} ln\ u_i(x)}{Z_{1-\alpha}} \right)^2, \tag{2.3.18}$$

$$\sum_{i=1}^{n} \frac{1}{r_i} \leq \left(\frac{d - \sum\limits_{i=1}^{n} ln\ u_i(x)}{Z_{\beta}} \right)^2. \tag{2.3.19}$$

Let $M = \min \left\{ \left(\frac{d - \sum\limits_{i=1}^{n} ln\ u_i(x)}{Z_{1-\alpha}} \right)^2, \left(\frac{d - \sum\limits_{i=1}^{n} ln\ u_i(x)}{Z_{\beta}} \right)^2 \right\}$.

Then the above optimization problem can be rewritten as

$$\min_{r_i} C = \sum_{i=1}^{n} c_i r_i$$

such that

$$\sum_{i=1}^{n} \frac{1}{r_i} \leq M, \tag{2.3.20}$$

where $d \in [ln(1 - R_1 + \epsilon),\ ln(1 - R_0 - \epsilon)]$, and the optimum value of d can be found by Lemma 2.5.3. This is an integer programming problem and can easily be solved.

Case 2: If $\sum\limits_{i=1}^{n} ln\ u_i(x) < ln(1 - R_1 + \epsilon)$

$$\min_{r_i} C = \sum_{i=1}^{n} c_i r_i$$

such that

$$\sum_{i=1}^{n} \frac{1}{r_i} \leq \left(\frac{d - ln(1 - R_1 + \epsilon)}{Z_{1-\alpha}} \right)^2, \tag{2.3.21}$$

$$\sum_{i=1}^{n} \frac{1}{r_i} \leq \left(\frac{d - \sum\limits_{i=1}^{n} ln\ u_i(x)}{Z_{\beta}} \right)^2. \tag{2.3.22}$$

Let $M' = \min\left\{ \left(\frac{d - ln(1-R_1+\epsilon)}{Z_{1-\alpha}}\right)^2, \left(\frac{d - \sum\limits_{i=1}^{n} ln\ u_i(x)}{Z_\beta}\right)^2 \right\}$.

Then the above optimization problem can be rewritten as

$$\min_{r_i} C = \sum_{i=1}^{n} c_i r_i$$

such that

$$\sum_{i=1}^{n} \frac{1}{r_i} \leq M', \tag{2.3.23}$$

where $d \in [ln(1-R_1+\varepsilon),\ ln(1-R_0-\varepsilon)]$, and the optimum value of d can be found by Lemma 2.5.3. This is also an integer programming problem and can easily be solved.

2.3.6 An algorithm to solve Problem Q

Let n be the number of different components in the parallel system, k be the number of covariates, x_j be the value of the covariate, c_i be the cost of testing the $i-th$ component. Let C denote the optimum cost and r_i denote the optimum number of components of type i to be tested for failures. The algorithm is described in following steps.

Step 1: Input the values for $n,\ k,\ x_j,\ c_i,\ R_0,\ R_1,\ \varepsilon,\ \alpha,\ \beta$ and $u_i(x)$ for $i = 1, 2, ..., n$ and $j = 1, 2, ..., k$.

Step 2: $M_1 = max\left\{ \left(\frac{ln(1-R_1+\varepsilon) - \sum\limits_{i=1}^{n} lnu_i(x)}{Z_{1-\alpha}}\right)^2, \left(\frac{ln(1-R_0-\varepsilon) - \sum\limits_{i=1}^{n} lnu_i(x)}{Z_{1-\alpha}}\right)^2 \right\}$.

Step 3: $M_2 = max\left\{ \left(\frac{ln(1-R_1+\varepsilon) - \sum\limits_{i=1}^{n} lnu_i(x)}{Z_\beta}\right)^2, \left(\frac{ln(1-R_0-\varepsilon) - \sum\limits_{i=1}^{n} lnu_i(x)}{Z_\beta}\right)^2 \right\}$.

Step 4: $M_3 = \left(\frac{ln(1-R_0-\varepsilon) - ln(1-R_1+\varepsilon)}{Z_{1-\alpha}}\right)^2$.

Step 5: If $\sum\limits_{i=1}^{n} ln\left(u_i(x)\right) \geq ln(1-R_1+\varepsilon)$, then go to Step 6. Else go to step 8.

Step 6: $M = min\left\{M_1, M_2\right\}$.

Step 7: Solve the integer programming problem

$$\min_{r_i} C = \sum_{i=1}^{n} c_i r_i$$

such that

$$\sum_{i=1}^{n} \frac{1}{r_i} \leq M,$$

r_i's are positive integers.

Step 8: If $\sum_{i=1}^{n} ln\,(u_i(x)) < ln(1 - R_1 + \varepsilon)$, then go to next step.

Step 9: $M' = min\,(M_3, M_2)$.

Step 10: Solve the integer programming problem

$$\min_{r_i} C = \sum_{i=1}^{n} c_i r_i$$

such that

$$\sum_{i=1}^{n} \frac{1}{r_i} \leq M',$$

r_i's are positive integers.

Step 11: Display the optimum cost C.

Step 12: Display $r_i \; \forall \; i = 1, 2, ..., n$.

Step 13: Exit

2.3.7 Numerical results and discussion

In this section, the Problem Q using algorithm designed in Section 2.3.6 is solved. The numerical results for 10 component parallel system with 3 covariates are presented. Let $\mathbf{X}$ denote the covariate vector, $\mathbf{c}$ denote the cost vector and $\mathbf{u}$ denote the upper bound vector of failure rates.

Example 2.3.7.1:

Let $R_0 = 0.95,\; R_1 = 0.999,\; \varepsilon = 0.0001,\; \alpha = 0.0001,\; \beta = 0.0001$ and the vectors $\mathbf{X} = (6, 8, 5)$, $\mathbf{c} = (8, 10, 5, 7, 12, 23, 31, 42, 54, 20)$ and $\mathbf{u} = (u_1, u_2, u_3, u_4,\; u_5, u_6, u_7, u_8, u_9, u_{10})$, where

$$u_1 = \frac{x_1}{x_1 + x_2 + x_3},\; u_2 = \frac{x_2}{2(x_1 + x_2 + x_3)},\; u_3 = \frac{x_3}{0.75(x_1 + x_2 + x_3)},\; u_4 = \frac{x_1}{x_1 + x_2 + x_3},$$

$$u_5 = \frac{x_2}{2(x_1 + x_2 + x_3)},\; u_6 = \frac{x_3}{0.8(x_1 + x_2 + x_3)},\; u_7 = \frac{x_1}{1.1(x_1 + x_2 + x_3)},$$

$$u_8 = \frac{x_2}{1.9(x_1 + x_2 + x_3)},\; u_9 = \frac{x_3}{0.7(x_1 + x_2 + x_3)},\; u_{10} = \frac{x_1}{(x_1 + x_2 + x_3)}.$$

For this set of data, the optimum sample size vector is $\mathbf{r} = (14, 12, 18, 15, 12, 8, 7, 6, 5, 9)$ and the corresponding optimum cost is $C = 1674$ units.

Example 2.3.7.2:

In Example 1, if we take $\mathbf{X} = (15, 29, 20)$ and $R_0 = 0.8$, then the corresponding optimum sample size vector and cost are $\mathbf{r} = (1, 1, 2, 1, 1, 1, 1, 1, 1, 1)$ and $C = 217$ units respectively.

The following Table 2.2 gives the optimum cost, corresponding sample sizes and a comparison of optimum costs with that in paper [72]. Let $\varepsilon = 0.0001$, the covariate vector $\mathbf{X} = (6,\ 8,\ 5)$ and the upper bound vector of the failure rates $\mathbf{u} = \left(\dfrac{x_1}{x_1+x_2+x_3}, \dfrac{x_2}{2(x_1+x_2+x_3)}\right)$. r_1^*, r_2^* and C^* denote the optimum sample sizes and optimum cost respectively obtained by proposed plan, and r_1, r_2 and C denote the optimum sample sizes and optimum cost respectively obtained by the plan in paper [72]. Let %CR denote the percentage of cost reduction of the proposed plan as compared with the plan obtained in paper [72]. Note that, for example 6 in Table 2.2, there is about 70% cost reduction is available for the plan developed using MLE approach.

Table 2.2: Numerical examples for two component system and comparison of results

	α	β	R_0	R_1	c_1	c_2	r_1^*	r_2^*	C^*	r_1	r_2	C	$\%CR$
1	0.05	0.05	0.96	0.99	1	1	2	2	4	3	3	6	33.33
2	0.05	0.05	0.95	0.99	1	1	2	2	4	2	3	5	20
3	0.02	0.02	0.94	0.99	1	1	3	2	5	2	4	6	16.66
4	0.05	0.1	0.99	0.999	1	1	1	1	2	1	2	3	33.33
5	0.05	0.05	0.99	0.999	1	1	1	1	2	1	2	3	33.33
6	0.03	0.03	0.91	0.98	1.5	2	1	1	3.5	4	3	12	70.83
7	0.05	0.1	0.99	0.999	1.5	2	1	1	3.5	2	1	5	30
8	0.05	0.05	0.99	0.999	1.5	2	1	1	3.5	2	1	5	30

2.4 Conclusions

In this chapter, an optimal reliability test plan for a parallel system with failure rate as the exponential function of covariates is designed. Maximum likelihood estimator and unbiased estimator of failure rate are used to estimate system reliability. The data are obtained through Type-II censoring scheme. Several numerical examples are illustrated and compared with the existing results. It is observed that there is a significant reduction in testing costs of about 70% with the test plan obtained using maximum likelihood estimator of failure rate. Also, system reliability is estimated using unbiased estimator of failure rate; and in this case, an optimal reliability test plan is obtained, as well. It is observed that the plan $R(\alpha,\ \beta,\ R_0,\ R_1,\ \mathbf{c},\ \mathbf{X})$ obtained using unbiased estimation approach has an potential of reducing testing costs of about 60% as compared to that in [100].

2.5 Appendix

Lemma 2.5.1. *Let* $f(x) = \left(\dfrac{a - \sum\limits_{i=1}^{n} x_i}{b}\right)$ *is a linear function of* $x_i, i = 1, 2, ..., n,$ *where* $a < 0,\ b > 0,\ x_i < 0,\ x = (x_1, x_2, ..., x_n)$ *and* $f(x) > 0$*, then*

1. *If* $\sum\limits_{i=1}^{n} c_i \geq c$*, then the minimum value of* $f(x)$ *subject to the constraints* $\sum\limits_{i=1}^{n} x_i \leq c$ *and* $x_i \leq c_i\ \forall\ i = 1, 2, ..., n,$ *where* c *and* c_i*'s are negative constants, is attained at* $x_i = c_i\ \forall\ i = 1, 2, ..., n$*.*

2. *If* $\sum\limits_{i=1}^{n} c_i < c$*, then the minimum value of* $f(x)$ *subject to the constraints* $\sum\limits_{i=1}^{n} x_i \leq c$ *and* $x_i \leq c_i\ \forall\ i = 1, 2, ..., n,$ *where* c *and* c_i*'s are negative constants, is attained at* c*.*

Proof. 1. Since $\sum\limits_{i=1}^{n} c_i \geq c$, the constraint $\sum\limits_{i=1}^{n} x_i \leq c$ is above our feasible region. Therefore, it is enough to minimize $f(x) = \left(\dfrac{a - \sum\limits_{i=1}^{n} x_i}{b}\right)$ subject to $x_i \leq c_i\ \forall i = 1, 2, ..., n$. Since this is a convex linear programming problem, the optimum lies on the boundary and the optimum value of $f(x)$ is $\left(\dfrac{a - \sum\limits_{i=1}^{n} c_i}{b}\right)$.

2. Since the given objective function and the constraints are linear convex functions, the optimum lies on the boundary. Therefore, it is enough to find the minimum value of $f(x)$ subjected to the constraints $\sum\limits_{i=1}^{n} x_i = c$ and $x_i = c_i\ \forall i = 1, 2, ..., n$.

This is a linear minimization problem with equality constraints. Clearly, the optimum attains at any one of the $n+1$ corner points

$$C_{ij}^{p} = \begin{cases} c - \sum\limits_{j=1, j\neq i}^{n} c_j \quad , & \text{if } i = j \\ c_j \quad , & \text{if } i \neq j \end{cases}$$

Then, clearly the minimum value of $f(x)$ is $\left(\dfrac{a-c}{b}\right)$

□

Lemma 2.5.2. *Let* $f(x) = \left(\dfrac{a - \sum\limits_{i=1}^{n} x_i}{b}\right)$ *is a linear function of* $x_i, i = 1, 2, ..., n,$ *where* $a < 0,\ b > 0,\ x_i < 0,\ x = (x_1, x_2, ..., x_n)$ *and* $f(x) < 0$*. Then the maximum value of* $f(x)$ *subjected to the constraints* $\sum\limits_{i=1}^{n} x_i \geq c$ *and* $x_i \leq c_i\ \forall\ i = 1, 2, ..., n,$ *where* c *and* c_i*'s are negative constants, is attained at* $x_i = c_i\ \forall\ i = 1, 2, ..., n.$

Proof. Since the given problem is a linear convex maximization problem, the optimum value will be attained at corner points. Then, clearly the optimum value of $f(x)$ is $\left(\dfrac{a - \sum\limits_{i=1}^{n} c_i}{b}\right)$

□

Lemma 2.5.3. *Let* $f(x) = \left(\dfrac{x-a}{b}\right)^2$*, where* $a < 0,\ x = (x_1, x_2, ..., x_n)$ *and* $x \in [-x_0, -x_1];\ x_0, x_1 > 0$*. Then the maximum value of* $f(x)$ *is* M*, where*

$$M = \max\left\{\left(\frac{-x_0 - a}{b}\right)^2, \left(\frac{-x_1 - a}{b}\right)^2\right\}.$$

Proof. **Case 1:** $a < -x_0$.

We have $a < -x_0 \Rightarrow a < -x_1$.
Then $0 < -x_0 - a < -x_1 - a$
$\Rightarrow \dfrac{(-x_0 - a)^2}{b^2} < \dfrac{(-x_1 - a)^2}{b^2}$.
This implies the maximum value is $\dfrac{(-x_1 - a)^2}{b^2}$.

Case 2: $a > -x_1$.

We have $0 > -x_1 - a > -x_0 - a$
$\Rightarrow -x_0 - a < -x_1 - a < 0$
$\Rightarrow (-x_0 - a)^2 > (-x_1 - a)^2$
$\Rightarrow \dfrac{(-x_0 - a)^2}{b^2} > \dfrac{(-x_1 - a)^2}{b^2}$.
Therefore the maximum value is $\dfrac{(-x_0 - a)^2}{b^2}$.
Case 3: $-x_0 \leq a \leq -x_1$.

We have $-x_0 - a < 0 < -x_1 - a$
$\Rightarrow \dfrac{(-x_0 - a)^2}{b^2} > 0$ and $\dfrac{(-x_1 - a)^2}{b^2} > 0$
Define $M = \max\left\{\left(\dfrac{-x_0 - a}{b}\right)^2, \left(\dfrac{-x_1 - a}{b}\right)^2\right\}$.
Then, clearly the maximum attains at M. □

Chapter 3

Design of Optimal Reliability Test Plan for a Series system in the Presence of Covariates

3.1 Introduction

The fundamental objective of reliability testing is to evaluate the system reliability and to demonstrate that the system will perform satisfactorily, prior to its deployment to the concerned field. A reliability test plans have been constructed in the paper [12], for a series system with redundant subsystems. In paper [13], a reliability test plan for a series system is obtained by assuming a constant failure rate that depends upon the mission performed. Later in 2010, a reliability test plan is designed for a series system in the paper [73] under mixed censoring. In this work, they have shown that optimal sample size is the same for all components.

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*Some results of this chapter are published in the following papers.

M. Kumar and P. N. Bajeel.:*Design of component reliability test plan for a series system having time dependent testing cost with the presence of covariates.* Computational Statistics. Springer-Verlag Germany. Vol. 33 (3), pp. 1267–1292, (2018) - (SCI & Scopus indexed).

P. N. Bajeel and M. Kumar.:*Reliability test plan for a series system with variable failure rates.* International Journal of Quality & Reliability Management. Emerald Publishing Limited. Vol. 34 (6), pp. 849–861, (2017) - (Scopus indexed).

M. Kumar and P. N. Bajeel.:*Design of optimal reliability test plans for series system in the presence of covariates.* Proceedings of KSCSTE, DST Sponsored International Conference on Advances in Applied Probability, Graph Theory and Fuzzy Mathematics, held at St. Peters college, Kollenchery, Jan. 11-14. ISBN:978-93-5174-243-2. pp. 51–61, (2014).

Also it is noted that authors in paper [82] have shown that, under Type-II censoring scheme, each component has to be tested equally, and the optimal common sample size does not depend upon the number of components in the system, and hence, testing an n component system is equivalent to testing a single component system. Here, they assume that the failure rates are constant but unknown. It is to be observed that in a realistic situation, this is not true since failure rate need not be a constant, instead, it is affected by some environmental factors like temperature, humidity, etc. called covariates. In literature, the design of test plans based on Type-II censoring (see, [73, 82]) consider the testing cost as a constant quantity that does not depend on time. It is to be noted that, in realistic situations, this is not true since the cost of testing in general, is a function of time under Type-II censoring. However, no test plans have been addressed in the literature that uses data obtained through Type-II censoring, and reports that the optimal sample size for each component is different for testing reliability of a series system.

In this chapter, the optimum design of component reliability test plan for a series system is designed in the presence of covariates. In Section 3.2, reliability test plan is constructed by assuming a normal testing cost and reliability is estimated using unbiased estimator of failure rate of the components. This Section 3.2 organized in section wise as follows: In Section 3.2.1, the problem is formulated. Section 3.2.2 describes the solution procedure for the optimization problem. In Section 3.2.3, the developed optimal test plan is illustrated through examples, and the optimal testing cost under the plan is compared with the costs associated with the plans available in literature [82]. In Section 3.3, the maximum-expected-testing-cost of the system is considered, and the system reliability is estimated using maximum likelihood estimator of failure rates, and thereby corresponding optimal reliability test plan is developed. This Section 3.3 organized in section wise as follows: Section 3.3.1 presents some preliminaries and formulation of the problem. Section 3.3.2 describes the solution procedure of the optimization problem stated in Section 3.3.1 and in Subsection 3.3.3, an algorithm is developed to solve the integer optimization problem formulated in Section 3.3.2. In Section 3.3.4, proposed optimal test plans are illustrated through examples. When testing cost is fixed (constant), the proposed plan is compared with the test plans obtained in [82] and [73]. It is observed that the proposed plan has huge potential to reduce the total number of components to be tested for failure, as compared to that obtained in [82] and [73]. A sensitivity analysis, simulation study, and qualitative analysis are also presented in Section 3.3.7 and 3.3.6 respectively. The conclusions are drawn in Section 3.4. Section 3.5 lists the appendix containing the statement of lemmas and their proofs.

3.2 Component reliability test plan with constant testing cost: Unbiased estimation approach

In this section, an optimal reliability test plan is designed for a series system having n different components by assuming that the lifetime of each component follows exponential distribution with variable failure rate as a parameter. That is, the failure rate as a function of environmental factors, such as temperature, pressure etc is considered. Type-II censoring scheme is used to obtain data. The decision rule for accepting the system is based on reliability estimate of the system. That is, the acceptance criteria based on system reliability estimate obtained using data from Type-II censoring.

3.2.1 Problem description

Let X_i denote the lifetime of $i-th$ component in series system, $1 \leq i \leq n$. Assume $X_i \sim Exp(\lambda_i(x))$, and are independent. The parameters $\lambda_i(x)$, of the exponential distributions depend upon k environmental factors ($x = (x_1, x_2, ..., x_k)$) through linear relationships. Here, it is assumed that $x_j > 0$ for $1 \leq j \leq k$. Therefore, $\lambda_i(x)$ can be written as

$$\lambda_i(x) = \beta_i x^T = \sum_{j=1}^{k} \beta_{ij} x_j = \beta_{i1} x_1 + \beta_{i2} x_2 + ... + \beta_{ik} x_k,$$

where $x = (x_1, x_2, ..., x_k)$, $\beta_i = (\beta_{i1}, \beta_{i2}, ..., \beta_{ik})$, $\beta_{ij} \in \mathbb{R}, i = 1, 2, ..., n; j = 1, 2, ..., k$. Let $u_i(x)$ be the upper bound for the i-th failure rate. Since $\lambda_i(x)$ is the parameter of exponential distribution, $\sum\limits_{j=1}^{k} \beta_{ij} x_j > 0 \;\; \forall \;\; i = 1, 2, ..., n$.

The i-th component is tested r_i times for failure and the lifetime $X_{ij}, 1 \leq i \leq n, 1 \leq$

$j \leq r_i$ is observed. Then $T_i = \sum\limits_{j=1}^{r_i} X_{ij}$ is a random variable that follows a Gamma distribution with shape parameter r_i and scale parameter $\lambda_i(x)$ and the expected value of T_i is given by $\dfrac{r_i}{\lambda_i(x)}$. Since T_i is a Gamma random variable, the expected value of $\dfrac{1}{T_i}$ and $\dfrac{1}{T_i^2}$ are $\dfrac{\lambda_i(x)}{r_i - 1}$ and $\dfrac{\lambda_i^2(x)}{(r_i-1)(r_i-2)}$ respectively. Therefore, the variance of $\dfrac{1}{T_i}$ is given by $\dfrac{\lambda_i^2(x)}{(r_i-1)^2(r_i-2)}$. As $E\left[\frac{1}{T_i}\right] = \lambda_i(x)\left(\frac{1}{r_i-1}\right)$, $(r_i-1)E\left[\frac{1}{T_i}\right] = \lambda_i(x)$. Thus, the quantity $\frac{r_i-1}{T_i}$ is an unbiased estimator of $\lambda_i(x)$.

Since the system being considered is a series system, the system reliability R for unit time period is given by $e^{-\sum\limits_{i=1}^{n}\lambda_i(x)}$. Using the unbiased estimator of failure rates, an estimator of system reliability for unit time is given by $\hat{R} = e^{-\sum\limits_{i=1}^{n}\frac{r_i-1}{T_i}}$. If $\hat{R} \geq d$, then the system is accepted, otherwise it is rejected. Note that $d \in (0,1)$. That is,

$$\hat{R} \geq d \Leftrightarrow e^{-\sum\limits_{i=1}^{n}\frac{r_i-1}{T_i}} \geq d \Leftrightarrow \sum_{i=1}^{n}\frac{r_i-1}{T_i} \leq -ln\ d.$$

Hence, the system is accepted iff $\sum\limits_{i=1}^{n}\dfrac{r_i-1}{T_i} \leq -ln\ d$, otherwise it is rejected.

A system is said to be good if R, the reliability of the system for unit time is greater than or equal to R_1, the acceptable reliability level. Then,

$$R \geq R_1 \Leftrightarrow \sum_{i=1}^{n}\lambda_i(x) \leq -lnR_1.$$

A system is said to be bad if R is less than or equal to R_0, the unacceptable reliability level. Then,

$$R \leq R_0 \Leftrightarrow \sum_{i=1}^{n}\lambda_i(x) \geq -lnR_0.$$

The reliability levels R_0 and R_1 are constants such that $0 < R_0 < R_1 < 1$.

Let the testing cost of the i-th component be c_i and total testing cost of the system be $C(\mathbf{r}) = \sum\limits_{i=1}^{n} c_i r_i$, where $\mathbf{r} = (r_1, r_2, ..., r_n)$. Similar type of testing cost expression is used in [82]. Then, based on the above acceptance rule, we have the following optimization problem:

$$\text{Minimize } C(\mathbf{r})$$

such that

$$P(\textit{Accept the system} \mid \textit{System is good}) \geq 1-\alpha, \tag{3.2.1}$$

$$P(\textit{Accept the system} \mid \textit{System is bad}) \leq \beta, \tag{3.2.2}$$

where $0 < \beta,\ 1-\alpha < 1$. The α and β in inequalities 3.2.1 and 3.2.2 are respectively referred to as producer's risk and consumer's risk.

3.2.2 Optimal design and solution

Using the acceptance rule defined in Section 3.2.1, the above formulated optimization problem can be written as follows:

$$\text{Minimize } C(\mathbf{r})$$

such that

$$P\left(\sum_{i=1}^{n}\frac{r_i-1}{T_i} \leq -ln\ d \mid \sum_{i=1}^{n}\sum_{j=1}^{k}\beta_{ij}x_j \leq -lnR_1\right) \geq 1-\alpha, \tag{3.2.3}$$

$$P\left(\sum_{i=1}^{n}\frac{r_i-1}{T_i} \leq -ln\ d \mid \sum_{i=1}^{n}\sum_{j=1}^{k}\beta_{ij}x_j \geq -lnR_0\right) \leq \beta, \tag{3.2.4}$$

$$r_i > 0 \text{ and integer } \forall\ \ i = 1,2,...,n, \quad \beta_{ij} \in \mathbb{R}.$$

The constraints 3.2.3 and 3.2.4 can be rewritten as

$$\min_{\beta_{ij}} P\left(\sum_{i=1}^{n}\frac{r_i-1}{T_i} \leq -ln\ d \mid \sum_{i=1}^{n}\sum_{j=1}^{k}\beta_{ij}x_j \leq -lnR_1\right) \geq 1-\alpha, \tag{3.2.5}$$

$$\max_{\beta_{ij}} P\left(\sum_{i=1}^{n}\frac{r_i-1}{T_i} \leq -ln\ d \mid \sum_{i=1}^{n}\sum_{j=1}^{k}\beta_{ij}x_j \geq -lnR_0\right) \leq \beta. \tag{3.2.6}$$

The exact distribution of $\sum_{i=1}^{n}\frac{r_i-1}{T_i}$ is not easy to obtain, and hence to make the problem tractable, an approximate distribution of $\sum_{i=1}^{n}\frac{r_i-1}{T_i}$ is obtained in the following. As explained in Section 3.2.1, note that $E\left(\frac{r_i-1}{T_i}\right) = \lambda_i(x)$ and $Var\left(\frac{r_i-1}{T_i}\right) = \frac{(\lambda_i(x))^2}{r_i-2}$.

It is further assumed that r_i is greater than 2. When n is relatively large, the Lindeberg Central Limit Theorem is invoked, so that the distribution of the test statistic $\sum\limits_{i=1}^{n}\frac{r_i-1}{T_i}$ is approximately normal with mean $\sum\limits_{i=1}^{n}\lambda_i(x)$ and variance $\sum\limits_{i=1}^{n}\frac{(\lambda_i(x))^2}{r_i-2}$. Therefore, by using the property of cumulative distribution function of normal random variable, the constraints 3.2.5 and 3.2.6 can be written as

$$\min_{\beta_{ij}}\left(\frac{-ln\ d-\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{k}\beta_{ij}x_j}{\sqrt{\sum\limits_{i=1}^{n}\frac{\left(\sum\limits_{j=1}^{k}\beta_{ij}x_j\right)^2}{r_i-2}}}\mid\sum_{i=1}^{n}\sum_{j=1}^{k}\beta_{ij}x_j\leq -lnR_1\right)\geq Z_{1-\alpha}, \tag{3.2.7}$$

$$\max_{\beta_{ij}}\left(\frac{-ln\ d-\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{k}\beta_{ij}x_j}{\sqrt{\sum\limits_{i=1}^{n}\frac{\left(\sum\limits_{j=1}^{k}\beta_{ij}x_j\right)^2}{r_i-2}}}\mid\sum_{i=1}^{n}\sum_{j=1}^{k}\beta_{ij}x_j\geq -lnR_0\right)\leq Z_{\beta}, \tag{3.2.8}$$

where, $Z=\frac{\sum\limits_{i=1}^{n}\frac{r_i-1}{T_i}-\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{k}\beta_{ij}x_j}{\sqrt{\sum\limits_{i=1}^{n}\frac{\left(\sum\limits_{j=1}^{k}\beta_{ij}x_j\right)^2}{r_i-2}}}$. To solve the optimization problems given by the inequalities 3.2.7 and 3.2.8, we invoke Lemma 3.5.3 (see Section 3.4 for statement and proof). Therefore, by Lemma 3.5.3, it is sufficient to solve the optimization problem on the boundary of feasible region. Hence 3.2.7 and 3.2.8 can be rewritten as

$$\min_{\beta_{ij}}\left(\frac{-ln\ d+lnR_1}{\sqrt{\sum\limits_{i=1}^{n}\frac{\left(\sum\limits_{j=1}^{k}\beta_{ij}x_j\right)^2}{r_i-2}}}\mid\sum_{i=1}^{n}\sum_{j=1}^{k}\beta_{ij}x_j= -lnR_1\right)\geq Z_{1-\alpha}, \tag{3.2.9}$$

$$\max_{\beta_{ij}} \left(\frac{-ln\ d + lnR_0}{\sqrt{\sum\limits_{i=1}^{n} \frac{\left(\sum\limits_{j=1}^{k} \beta_{ij}x_j\right)^2}{r_i - 2}}} \mid \sum_{i=1}^{n}\sum_{j=1}^{k} \beta_{ij}x_j = -lnR_0 \right) \leq Z_\beta. \tag{3.2.10}$$

Note that for $\alpha,\ \beta < 0.5,\ Z_{1-\alpha} > 0$ and $Z_\beta < 0$. This implies $-ln\ d \in (-ln\ R_1, -ln\ R_0)$.

Consider the optimization problem represented by left hand side (LHS) of the constraint 3.2.9. For all $\alpha < 0.5,\ Z_{1-\alpha} > 0$. Therefore, the optimum value of this optimization problem is positive. To find the minimum value of the objective function, it is enough to maximize the denominator. Hence, we have

Problem $\mathcal{P}(1)$

$$\text{Maximize } \sum_{i=1}^{n} \frac{\left(\sum\limits_{j=1}^{k} \beta_{ij}x_j\right)^2}{r_i - 2}$$

such that

$$\sum_{i=1}^{n}\sum_{j=1}^{k} \beta_{ij}x_j = -lnR_1,$$

$$r_i > 2\ \forall\ \ i = 1, 2, ...n.$$

Problem $\mathcal{P}(1)$ is unbounded and has no solution. However, incorporating the priori information in the form of upper bound on failure rates, Problem $\mathcal{P}(1)$ can be restated as:

Problem $\mathcal{P}(2)$

$$\text{Maximize } \sum_{i=1}^{n} \frac{\left(\sum\limits_{j=1}^{k} \beta_{ij}x_j\right)^2}{r_i - 2}$$

such that

$$\sum_{i=1}^{n}\sum_{j=1}^{k} \beta_{ij}x_j = -lnR_1,$$

$$\sum_{j=1}^{k} \beta_{ij}x_j \leq u_i(x)\ \forall\ i = 1, 2, ..., n,$$

$$r_i > 2 \ \forall \ \ i = 1, 2, ...n.$$

Let $x_{min} = Min\{x_1, x_2, ..., x_k\}$ and m is the value of j corresponding to x_{min}. Then for $1 \leq i \leq n, 1 \leq j \leq k$ and $1 \leq p \leq n$, define

$$\mu_{ij}^p = \begin{cases} \frac{u_i(x)}{x_j}, & \text{if } \ (i \neq p) \text{ and } \ (j = m) \\ \\ \frac{-lnR_1 - \sum\limits_{i \neq p} u_i(x)}{x_{min}}, & \text{if } \ (i = p) \text{ and } \ (j = m) \\ \\ 0, & \text{elsewhere.} \end{cases}$$

Then, as per Lemma 3.5.4 (see Section 3.4 for statement and proof), the optimum solution to Problem $\mathcal{P}(2)$ will be at any one of these μ_{ij}^p.

Consider the optimization problem represented by LHS of constraint 3.2.10. For all $\beta < 0.5$, $Z_\beta < 0$. Therefore, the optimum value of this problem is negative and hence to find the maximum value of the objective function, it is enough to maximize the denominator. That is, we have

Problem $\mathcal{P}(3)$

$$\text{Maximize} \sum_{i=1}^{n} \frac{\left(\sum\limits_{j=1}^{k} \beta_{ij} x_j\right)^2}{r_i - 2}$$

such that

$$\sum_{i=1}^{n} \sum_{j=1}^{k} \beta_{ij} x_j = -lnR_0,$$

$$r_i > 2 \ \forall \ \ i = 1, 2, ...n.$$

Observe that $\mathcal{P}(3)$ is an unbounded optimization problem and has no solution. Incorporation of the priori information in the form of upper bound on failure rates, Problem $\mathcal{P}(3)$ can be redefined as:

Problem $\mathcal{P}(4)$

$$\text{Maximize} \sum_{i=1}^{n} \frac{\left(\sum\limits_{j=1}^{k} \beta_{ij} x_j\right)^2}{r_i - 2}$$

such that

$$\sum_{i=1}^{n}\sum_{j=1}^{k}\beta_{ij}x_j = -lnR_0,$$

$$\sum_{j=1}^{k}\beta_{ij}x_j \leq u_i(x)\ \forall\ i = 1, 2, ..., n,$$

$$r_i > 2\ \forall\ \ i = 1, 2, ...n.$$

Now, for $1 \leq i \leq n, 1 \leq j \leq k$ and $1 \leq p \leq n$, define

$$\gamma_{ij}^{p} = \begin{cases} \frac{u_i(x)}{x_j}, & \text{if } (i \neq p) \text{ and } (j = m) \\ \frac{-lnR_0 - \sum\limits_{i \neq p} u_i(x)}{x_{min}}, & \text{if } (i = p) \text{ and } (j = m) \\ 0, & \text{elsewhere.} \end{cases}$$

Then by Lemma 3.5.4, the optimum solution to Problem $\mathcal{P}(4)$ will be at any one of these γ_{ij}^{p}.

Observe that for Problem $\mathcal{P}(2)$ and Problem $\mathcal{P}(4)$ to be feasible, it is necessary that $\sum\limits_{i=1}^{n} u_i(x) \geq -lnR_0$.

Let the optimum values of Problem $\mathcal{P}(2)$ and Problem $\mathcal{P}(4)$ be μ_{ij}^{p*} and γ_{ij}^{p*} respectively. Then the optimization problem with constraints 3.2.9 and 3.2.10 can be written as:

Problem $\mathcal{P}(5)$

Minimize $C(\mathbf{r})$

such that

$$\sum_{i=1}^{n}\frac{\left(\sum\limits_{j=1}^{k}\mu_{ij}^{p*}x_j\right)^2}{r_i - 2} \leq \left(\frac{-ln\ d + lnR_1}{Z_{1-\alpha}}\right)^2,$$

$$\sum_{i=1}^{n} \frac{\left(\sum_{j=1}^{k} \gamma_{ij}^{p*} x_j\right)^2}{r_i - 2} \leq \left(\frac{-ln\ d + lnR_0}{Z_\beta}\right)^2,$$

$$r_i > 2 \text{ and integer } \forall\ \ i = 1, 2, ..., n,\ \beta_{ij} \in \mathbb{R},$$

provided

$$\sum_{i=1}^{n} \sum_{j=1}^{k} \gamma_{ij}^{p*} x_j + ln\ d < \left(\frac{r_i - 2}{\sum_{j=1}^{k} \gamma_{ij}^{p*} x_j}\right) \left(\sum_{i=1}^{n} \frac{\left(\sum_{j=1}^{k} \gamma_{ij}^{p*} x_j\right)^2}{r_i - 2}\right),$$

for all $i = 1, 2, ..., n$, where $-ln\ d \in (-lnR_1, -lnR_0)$.

The problem $\mathcal{P}(5)$ is solved with the help of LINGO 11 software and Visual C++.

3.2.3 Numerical Results and Comparison

In this section, some numerical results to illustrate the test plan developed in Section 3.2.2 are presented. Further, these results are compared with those results obtained by Rajgopal and Mazumdar (1996: [82]). For this comparison purpose, let $n = 6$, $k = 3$, $R_0 = 0.8$ and $R_1 = 0.99$. Let $\mathbf{X}$ denotes the vector of environmental factors. The common sample size and the optimal cost obtained through the test plan described in [82] are denoted respectively by r^* and z^*. Parameters r_i^*, d^* and z_c^* denote respectively the optimal sample size of the i-th component, the optimal value of d and the optimal cost as per the test plan developed in Section 3.3.2. Table 3.1 gives the optimum costs and corresponding sample sizes.

Table 3.1: Numerical examples for six component system and comparison of results.

Parameters	Example 1	Example 2	Example 3	Example 4
α	0.001	0.001	0.0001	0.0001
β	0.001	0.001	0.0001	0.0001
X	4, 60, 21	11, 39, 6	4, 60, 21	15, 22, 5
c_1	7	100	7	1
c_2	2	200	2	2
c_3	8.3	250	8.3	3.5
c_4	6	1500	6	5
c_5	11	2500	11	8
c_6	4.5	3000	4.5	4
u_1	$\frac{x_1}{65(x_1+x_2)}$	$\frac{x_1}{9(x_1+x_2)}$	$\frac{x_1}{65(x_1+x_2)}$	$\frac{x_1}{9x_1+10x_2}$
u_2	$\frac{x_2}{4(x_1+x_2)}$	$\frac{x_2}{4(x_1+x_2)}$	$\frac{x_2}{4(x_1+x_2)}$	$\frac{x_2}{2x_1+4x_2}$
u_3	$\frac{x_3}{59(x_1+x_2)}$	$\frac{x_3}{50(x_1+x_2)}$	$\frac{x_3}{59(x_1+x_2)}$	$\frac{x_3}{50x_1+55x_2}$
u_4	$\frac{1}{67(x_1+x_2)}$	$\frac{1}{51(x_1+x_2)}$	$\frac{1}{67(x_1+x_2)}$	$\frac{1}{51x_1+60x_2}$
u_5	$\frac{1}{70(x_1+x_2)}$	$\frac{1}{57(x_1+x_2)}$	$\frac{1}{70(x_1+x_2)}$	$\frac{1}{57x_1+63x_2}$
u_6	$\frac{1}{63(x_1+x_3)}$	$\frac{1}{55(x_1+x_3)}$	$\frac{1}{63(x_1+x_3)}$	$\frac{1}{59x_1+55x_3}$
r_1^*	3	5	3	11
r_2^*	18	22	24	29
r_3^*	3	3	3	3
r_4^*	3	3	3	3
r_5^*	3	3	3	3
r_6^*	3	3	3	3
z_c^*	146.5	26650	158.4	130.5
d^*	0.050325	0.0799449	0.0471286	0.0856984
r^*	14	14	19	19
z^*	543.2	105700	737.2	446.5
(%) Cost reduction	73.04 %	74.78 %	78.51 %	70.77 %

Consider the following two examples of a series system with 9 components and 3 environmental factors (covariates).

Example 3.2.3.1:

$R_0 = 0.8,\ R_1 = 0.99,\ \alpha = \beta = 0.0001,\ X = (15, 22, 5),\ C = (1, 2, 3, 5, 8, 4, 6, 7, 9)$, $u_1 = \dfrac{x_1}{2(x_1+x_2)},\ u_2 = \dfrac{x_2}{91(x_1+x_2)},\ u_3 = \dfrac{x_3}{80(x_1+x_2)},\ u_4 = \dfrac{1}{87(x_1+x_2)},\ u_5 = \dfrac{x_1}{82(x_1+x_2)},\ u_6 = \dfrac{x_2}{90.5(x_1+x_3)},\ u_7 = \dfrac{1}{92.5(x_2+x_3)},\ u_8 = \dfrac{1}{83.5(x_1+x_2+x_3)}$,

$u_9 = \dfrac{x_1}{93.5(x_1 + x_2 + x_3)}$. Then the optimum values of r_i^*'s are $r_1^* = 30, r_2^* = 3, r_3^* = 3, r_4^* = 3, r_5^* = 3, r_6^* = 3, r_7^* = 3, r_8^* = 3, r_9^* = 3$ and $z_c^* = 162, d^* = 0.0738664$. Comparing the results obtained by using our plan with those obtained by Rajgopal and Mazumdar (1996: [82]) in which the common sample size is 19 and the optimum cost is 855, it is seen that the optimum cost is reduced by about 81%.

Example 3.2.3.2:

$R_0 = 0.8$, $R_1 = 0.99$, $\alpha = \beta = 0.0001$, $X = (9, 29, 4)$, $C = (12, 23, 31, 42, 54, 60, 71, 85, 99)$, $u_1 = \dfrac{x_2}{4(x_1 + x_2)}$, $u_2 = \dfrac{x_2}{91(x_2 + x_3)}$, $u_3 = \dfrac{x_1}{80(x_3 + x_1)}$, $u_4 = \dfrac{x_1}{87(x_1 + x_2)}$, $u_5 = \dfrac{x_1}{82(x_2 + x_3)}$, $u_6 = \dfrac{x_1}{90.5(x_3 + x_1)}$, $u_7 = \dfrac{x_3}{91(x_1 + x_2)}$, $u_8 = \dfrac{x_3}{91(x_2 + x_3)}$, $u_9 = \dfrac{x_3}{80(x_3 + x_1)}$. Then the optimum values of r_i^*'s are $r_1^* = 29, r_2^* = 4, r_3^* = 6, r_4^* = 3, r_5^* = 3, r_6^* = 3, r_7^* = 3, r_8^* = 3, r_9^* = 3$ and $z_c^* = 1859, d^* = 0.0862412$. In this case, it can be seen that total testing cost is reduced by about 78% as compared to the total cost involved in test plan obtained in [82].

3.3 Component reliability test plan with time dependent testing cost: MLE approach

Consider a series system having n different components. The aim is to design a reliability test plan for a series system, with n different components, having lifetime following Exponential distribution. The failure rates of components are assumed to be linear function of covariates. That is, it is assumed that the parameters $\lambda_i(\mathbf{x}), 1 \leq i \leq n$, of the exponential distributions depend upon covariate vector $\mathbf{x} \in \mathbb{R}^k$. All covariates are assumed to be greater than zero ($x_i > 0, 1 \leq i \leq k$). The dependence of $\lambda_i(\mathbf{x})$ on the k covariates ($x_i, i = 1, 2, ..., k$), is described using the linear function given by $\lambda_i(\mathbf{x}) = \sum_{j=1}^{k} \beta_{ij} x_j = \beta_{i1} x_1 + \beta_{i2} x_2 + ... + \beta_{ik} x_k$, $\beta_{ij} \in \mathbb{R}$, $i = 1, 2, ..., n; j = 1, 2, ..., k$. It is implicitly assumed throughout this section that β_{ij} and x_j are such that for $i = 1, 2, ..., n$, $\lambda_i(\mathbf{x}) = \sum_{j=1}^{k} \beta_{ij} x_j > 0$. The data is obtained through Type-II censoring, and reliability test plan is designed by considering the testing cost as a function of time, which is a random quantity. Observe that this is not considered so far in the literature, when designing reliability test plans for series system using data from

Type-II censoring.

3.3.1 Some Preliminaries and Problem Formulation

Consider a series system with n independent components. Let X_i denote the lifetime of $i-th$ component in the system, having Exponential distribution with parameters $\lambda_i(\mathbf{x})$, $1 \leq i \leq n$, respectively. Then the system reliability R for unit time period is given by $R = R_s(1) = exp\left(-\sum_{i=1}^{n} \lambda_i(\mathbf{x})\right)$. Test r_i identical components of type i to failure, and observe their lifetimes. In other words, during testing the component of type i, any failed component is replaced with an iid component. Let X_{ij} denote the lifetime of $j-th$ component of type i, $j = 1, 2, ..., r_i$. Then, it is clear that, for $1 \leq i \leq n$, $T_i = \sum_{j=1}^{r_i} X_{ij}$ is a random variable having Gamma distribution with shape parameter r_i and scale parameter $\lambda_i(\mathbf{x})$. The expected value and variance of $1/T_i$ are given by $\lambda_i(\mathbf{x})/(r_i - 1)$ and $(\lambda_i(\mathbf{x}))^2 / ((r_i - 1)^2(r_i - 2))$ respectively. Based on $(T_1, T_2, ..., T_n)$, the maximum likelihood estimator of $\lambda_i(\mathbf{x})$ is given by r_i/T_i, $1 \leq i \leq n$.

A series system at time $t = 1$, is considered satisfactory if $R \geq R_1$, and, it is considered to be unsatisfactory, if $R \leq R_0$, where R_0 and R_1 are constants such that $0 < R_0 < R_1 < 1$. Here, R_1 is called acceptable reliability level (ARL), R_0 is called unacceptable reliability level (URL). Note that $R \geq R_1 \Leftrightarrow \sum_{i=1}^{n} \lambda_i(\mathbf{x}) \leq -ln(R_1)$, and $R \leq R_0 \Leftrightarrow \sum_{i=1}^{n} \lambda_i(\mathbf{x}) \geq -ln(R_0)$.

Since a reliable system will tend to have a small value for the quantity $\sum_{i=1}^{n} \lambda_i(\mathbf{x})$, which is the sum of failure rates of n components, it is reasonable to accept the system when the MLE of system reliability $\hat{R} = exp\left(-\sum_{i=1}^{n} (r_i/T_i)\right)$ exceeds a value d in $(0, 1)$, which is to be determined later. Then,

$$\hat{R} \geq d \Leftrightarrow exp\left(-\sum_{i=1}^{n} (r_i/T_i)\right) \geq d \Leftrightarrow \sum_{i=1}^{n} (r_i/T_i) \leq -ln(d).$$

Therefore, accept the system iff $\sum_{i=1}^{n} (r_i/T_i) \leq -ln(d)$, otherwise reject it, where $d \in (0, 1)$.

Note that $T_i = \sum_{j=1}^{r_i} X_{ij}$ denote the total test duration of testing r_i components. Let $c_i(t)$ denote the cost of testing $i-th$ component for the time $t, 1 \leq i \leq n$. Then $c_i(1) = c_i$ denote cost of testing $i-th$ component for unit time. Now, $c_i(T_i)$ is the total cost of testing the component of type i for r_i failures. That is, c_iT_i is total testing cost for testing r_i components, which is random. Hence, the total-expected-testing-cost for $i-th$ component ($T^{(i)}ETC$) is given by $c_iE(T_i) = \frac{c_ir_i}{\lambda_i(x)}$. Therefore, the total-expected-testing-cost involved in testing n different components (i.e., for $i = 1, 2, ..., n$) is given by $TETC = \sum_{i=1}^{n} \frac{c_ir_i}{\lambda_i(x)}$, which is a function of $\lambda_i(x)$. Observe that $T^{(i)}ETC$ is a decreasing function of $\lambda_i(x)$. Since the total cost involved in testing is $\sum_{i=1}^{n} c_iT_i$, which is random, it is natural to consider total expected cost in testing. Hence, we propose to solve the following optimization Problem, $\mathcal{Q}_1$ based upon the acceptance rule defined above.

$$\text{Minimize } TETC = \sum_{i=1}^{n} \frac{c_ir_i}{\lambda_i(x)}$$

such that

$$P(\text{Accept the system} \mid \text{System is good}) \geq 1 - \alpha, \tag{3.3.1}$$

$$P(\text{Accept the system} \mid \text{System is bad}) \leq \beta, \tag{3.3.2}$$

where α and β are respectively the producer's and consumer's risk, $0 < \beta < 1-\alpha < 1$. Here, Inequality 3.3.1 indicates that the producer's risk is less than α, while the Inequality 3.3.2 indicates that the consumer's risk is at most β. Note that $TETC$ is the total expected cost of testing the entire series system.

3.3.2 Solution of the Problem $\mathcal{Q}_1$ and Optimal Design

Using the acceptance rule defined in Section 3.3.1, the Problem $\mathcal{Q}_1$, can be rewritten as Problem $\mathcal{Q}_2$ as follows:

$$\min_{r_i} TETC = \sum_{i=1}^{n} \frac{c_ir_i}{\lambda_i(x)}$$

such that

$$P\left(\sum_{i=1}^{n}(r_i/T_i) \le -ln(d) \mid R \ge R_1\right) \ge 1-\alpha,$$

$$P\left(\sum_{i=1}^{n}(r_i/T_i) \le -ln(d) \mid R \le R_0\right) \le \beta.$$

The above optimization problem is intractable since it involves the unknown $\lambda_i(x)$ in objective function. Hence proceed as follows: since the total expected cost is strictly a decreasing function of $\lambda_i(x)$, the minimum value of $\lambda_i(x)$ will give the maximum value of total-expected-testing-cost. Let it be λ_i^*. Note that $TETC$ is maximum for $\lambda_i(x) = \lambda_i^*$. Then maximum-total-expected-testing-cost ($MTETC$) will occur for $\lambda_i(x) = \lambda_i^*$. Let $T = \sum\limits_{i=1}^{n}(r_i/T_i)$ then, the following Problem $\mathcal{Q}_3$ which minimizes the upper bound for total-expected-testing-cost ($TETC$) is considered. In other words, we minimize the maximum expected cost involved in testing a series system.

$$\min_{r_i} MTETC = \sum_{i=1}^{n}\frac{c_i r_i}{\lambda_i^*}$$

such that

$$P\left(T \le -ln(d)\middle| \sum_{i=1}^{n}\sum_{j=1}^{k}\beta_{ij}x_j \le -ln(R_1), \sum_{j=1}^{k}\beta_{ij}x_j > 0 \;\; \forall \;\; i\right) \ge 1-\alpha, \tag{3.3.3}$$

$$P\left(T \le -ln(d)\middle| \sum_{i=1}^{n}\sum_{j=1}^{k}\beta_{ij}x_j \ge -ln(R_0), \sum_{j=1}^{k}\beta_{ij}x_j > 0 \;\; \forall \;\; i\right) \le \beta, \tag{3.3.4}$$

$r_i \ge 1,$ are integers $\forall \;\; i = 1, 2, ..., n,$ $\beta_{ij} \in \mathbb{R}$.

Now, in terms of probability of acceptance, Inequality 3.3.3, states that the probability of acceptance should be at least $1-\alpha$, for all $\lambda_i(\mathbf{x}) > 0$. That is, the minimum probability of acceptance over all such $\lambda_i(\mathbf{x})$ should exceed $1-\alpha$. Similarly, the Inequality 3.3.4 states that the probability of acceptance should be at most β, for all $\lambda_i(\mathbf{x}) > 0$. This means that, the maximum probability of acceptance over all such

$\lambda_i(\mathbf{x})$ should not exceed β. Therefore, Inequality 3.3.3 and 3.3.4 can be rewritten as

$$\min_{\beta_{ij}} P\left(T \leq -ln(d)\middle| \sum_{i=1}^{n}\sum_{j=1}^{k}\beta_{ij}x_j \leq -ln(R_1), \sum_{j=1}^{k}\beta_{ij}x_j > 0 \ \forall \ i\right) \geq 1-\alpha, \quad (3.3.5)$$

$$\max_{\beta_{ij}} P\left(T \leq -ln(d)\middle| \sum_{i=1}^{n}\sum_{j=1}^{k}\beta_{ij}x_j \geq -ln(R_0), \sum_{j=1}^{k}\beta_{ij}x_j > 0 \ \forall \ i\right) \leq \beta. \quad (3.3.6)$$

Note that the statistic T is a finite linear combination of independent Inverse gamma random variables $1/T_i,\ i = 1, 2, ..., n$. Although, the distribution function of T can be obtained by making use of inversion formula (see the paper, [95] for more detail), given by

$$P(T \leq s) = F(s) = \frac{1}{2} - \frac{1}{\pi}\int_0^{\infty}\left(\frac{e^{-its}\Phi_T(t) - e^{-its}\Phi_T(-t)}{2it}\right)dt, \quad (3.3.7)$$

where $\Phi_T(t)$ is the characteristic function of T. $F(s)$ cannot be evaluated analytically, and one has to resort to numerical integration techniques to approximate the probability for given value of s. However, this cannot be used to obtain the tractable optimization problem. Hence, we resort to apply well-known Delta method to get the distribution of T. Then $T = \sum_{i=1}^{n}(r_i/T_i)$, where T_i follows Gamma distribution with mean $\mu_i = r_i/\lambda_i(\mathbf{x})$ and variance $\sigma_i^2 = r_i/\left(\lambda_i(\mathbf{x})\right)^2$. A well-known Delta method can be used to get the approximate distribution of T (see, [83]). Note $T_i \sim N\left(\mu_i, \sigma_i^2\right)$ for all $i = 1, 2, ..., n$. Define $g(T_i) = r_i/T_i$, then $g'(T_i) = -1/T_i^2,\ g(\mu_i) = \lambda_i(\mathbf{x})$ and $g'(\mu_i) = -(\lambda_i(\mathbf{x}))^2/r_i$. Using the properties of asymptotic normality and efficiency of the MLE (see, [21,83]),

$$\sqrt{r_i}\left(\frac{r_i}{T_i} - \lambda_i(\mathbf{x})\right) \sim N(0, (\lambda_i(\mathbf{x}))^2).$$

Then by Delta method $g(T_i)$ follows Normal distribution with mean $g(\mu_i) = \lambda_i(\mathbf{x})$ and variance $\left(g'(\mu_i)\right)^2\sigma_i^2(T_i) = (\lambda_i(\mathbf{x}))^2/r_i$. That is,

$$g(T_i) \sim N\left(\lambda_i(\mathbf{x}), (\lambda_i(\mathbf{x}))^2/r_i\right).$$

By using Lindeberg central limit theorem, $T = \sum_{i=1}^{n} g(T_i) = \sum_{i=1}^{n}(r_i/T_i)$ follows Normal distribution with mean $\sum_{i=1}^{n}\lambda_i(\mathbf{x})$ and variance $\sum_{i=1}^{n}((\lambda_i(\mathbf{x}))^2/r_i)$. Then by using the property of cumulative distribution function of Normal distribution, the Inequalities 3.3.5 and 3.3.6 can be written as

$$\min_{\beta_{ij}}\left(\frac{-ln(d)-\sum_{i=1}^{n}\sum_{j=1}^{k}\beta_{ij}x_j}{\sqrt{\sum_{i=1}^{n}\left(\frac{\left(\sum_{j=1}^{k}\beta_{ij}x_j\right)^2}{r_i}\right)}}\middle|\sum_{i=1}^{n}\sum_{j=1}^{k}\beta_{ij}x_j\leq -ln(R_1),\ \sum_{j=1}^{k}\beta_{ij}x_j>0\ \forall\ i\right)\geq Z_{1-\alpha}, \tag{3.3.8}$$

$$\max_{\beta_{ij}}\left(\frac{-ln(d)-\sum_{i=1}^{n}\sum_{j=1}^{k}\beta_{ij}x_j}{\sqrt{\sum_{i=1}^{n}\left(\frac{\left(\sum_{j=1}^{k}\beta_{ij}x_j\right)^2}{r_i}\right)}}\middle|\sum_{i=1}^{n}\sum_{j=1}^{k}\beta_{ij}x_j\geq -ln(R_0),\sum_{j=1}^{k}\beta_{ij}x_j>0\ \forall\ i\right)\leq Z_{\beta}, \tag{3.3.9}$$

where, $Z=\frac{\sum_{i=1}^{n}(r_i/T_i)-\sum_{i=1}^{n}\sum_{j=1}^{k}\beta_{ij}x_j}{\sqrt{\sum_{i=1}^{n}\left(\frac{\left(\sum_{j=1}^{k}\beta_{ij}x_j\right)^2}{r_i}\right)}}$. In order to solve the optimization problem stated in 3.3.8 and 3.3.9, the following lemma is presented.

Lemma 3.3.1. *Let $f:\mathbb{R}^{nk}\longrightarrow\mathbb{R}$ be a function defined by*

$$f(\beta_{11},\beta_{12},...,\beta_{1k},...,\beta_{n1},\beta_{n2},...,\beta_{nk})=\frac{-ln(d)-\sum_{i=1}^{n}\sum_{j=1}^{k}\beta_{ij}x_j}{\sqrt{\sum_{i=1}^{n}\left(\frac{\left(\sum_{j=1}^{k}\beta_{ij}x_j\right)^2}{r_i}\right)}},$$

where $-ln(d)>0$, $d\in(0,1)$, $x_j>0$, $r_i\geq 1$, are integers, $\sum_{j=1}^{k}\beta_{ij}x_j>0\ \ \forall\ \ i$, and $\beta_{ij}\in\mathbb{R}$, $i=1,2,...,n$, $\ j=1,2,...,k$. Then the following statements are true.

a) If $-ln(d)\geq\sum_{i=1}^{n}\sum_{j=1}^{k}\beta_{ij}x_j$, then f is strictly decreasing function of $\beta_{ij}\ \ \forall\ \ i=1,2,...,n,\ \ j=1,2,...,k$.

b) If $-ln(d) < \sum_{i=1}^{n}\sum_{j=1}^{k}\beta_{ij}x_j$ *and*

$$\sum_{i=1}^{n}\sum_{j=1}^{k}\beta_{ij}x_j + ln(d) < \left(\frac{r_q}{\sum_{j=1}^{k}\beta_{qj}x_j}\right)\left(\sum_{i=1}^{n}\left(\frac{\left(\sum_{j=1}^{k}\beta_{ij}x_j\right)^2}{r_i}\right)\right),$$

then f *is strictly decreasing function of* β_{ij} $\forall\ i = 1,2,...,n;\ j = 1,2,...,k;\ q = 1,2,...,n.$

Proof: (see, Section 3.5.1).
By Lemma 3.3.1, it is sufficient to solve the optimization problem stated in Inequalities 3.3.8 and 3.3.9 on the boundary of feasible region. Hence, the Inequalities 3.3.8 and 3.3.8 can be rewritten as

$$\min_{\beta_{ij}}\left(\frac{-ln(d)+ln(R_1)}{\sqrt{\sum_{i=1}^{n}\left(\frac{\left(\sum_{j=1}^{k}\beta_{ij}x_j\right)^2}{r_i}\right)}}\middle|\sum_{i=1}^{n}\sum_{j=1}^{k}\beta_{ij}x_j = -ln(R_1), \sum_{j=1}^{k}\beta_{ij}x_j > 0\ \forall\ i\right) \geq Z_{1-\alpha}, \tag{3.3.10}$$

$$\max_{\beta_{ij}}\left(\frac{-ln(d)+ln(R_0)}{\sqrt{\sum_{i=1}^{n}\left(\frac{\left(\sum_{j=1}^{k}\beta_{ij}x_j\right)^2}{r_i}\right)}}\middle|\sum_{i=1}^{n}\sum_{j=1}^{k}\beta_{ij}x_j = -ln(R_0), \sum_{j=1}^{k}\beta_{ij}x_j > 0\ \forall\ i\right) \leq Z_{\beta}. \tag{3.3.11}$$

For α and β less than 0.5, $Z_{1-\alpha} > 0$ and $Z_\beta < 0$. This implies that $-ln(d) \in (-ln(R_1), -ln(R_0))$. Now, consider the optimization problem represented by LHS of the constraints 3.3.10 and 3.3.11. Since the optimum values of these problems are positive and negative respectively, the minimum and maximum are obtained by maximizing the denominators. That is, the LHS of 3.3.10 and 3.3.11, reduces to solving the following problems:

Problem $\mathcal{Q}_4$

$$\text{Maximize} \sum_{i=1}^{n} \left(\frac{\left(\sum_{j=1}^{k} \beta_{ij} x_j \right)^2}{r_i} \right)$$

such that

$$\sum_{i=1}^{n} \sum_{j=1}^{k} \beta_{ij} x_j = -ln(R_1)$$

$$\sum_{j=1}^{k} \beta_{ij} x_j > 0 \ \ \forall \ \ i = 1, 2, ..., n$$

$$r_i \geq 1, \text{ are integers } \forall \ \ i = 1, 2, ..., n$$

$$\beta_{ij} \in \mathbb{R}.$$

Problem $\mathcal{Q}_5$

$$\text{Maximize} \sum_{i=1}^{n} \left(\frac{\left(\sum_{j=1}^{k} \beta_{ij} x_j \right)^2}{r_i} \right)$$

such that

$$\sum_{i=1}^{n} \sum_{j=1}^{k} \beta_{ij} x_j = -ln(R_0)$$

$$\sum_{j=1}^{k} \beta_{ij} x_j > 0 \ \ \forall \ \ i = 1, 2, ..., n$$

$$r_i \geq 1, \text{ are integers } \forall \ \ i = 1, 2, ..., n$$

$$\beta_{ij} \in \mathbb{R}.$$

Examining the above problems, it is clear that by choosing some β_{ij} to be arbitrarily small, we can increase the objective function indefinitely and hence feasibility can always be maintained by making one or more of the other β_{ij} sufficiently large. Thus, from the mathematical perspective, Problem $\mathcal{Q}_4$ and Problem $\mathcal{Q}_5$ are unbounded problems and have no solutions. Also note that $R = exp\left(-\sum_{i=1}^{n} \lambda_i(\mathbf{x})\right)$, where all the $\lambda_i(\mathbf{x})$ are sufficiently small. While attempting to solve this unbounded problem, it may give a solution, where the assumption that failure rates are linear functions of covariates becomes invalid. Thus, the optimization problem itself is no longer meaningful.

Given the obvious desirability of an acceptance rule that is based on maximum likelihood estimation of failure rates, now examine a particular situation, where this rule can be used in practice. In many instances, it is realistic to assume that some priori information on component reliabilities available. If the component has been used as part of some other system or in an earlier model of the present system, one may have some knowledge of its failure rate. In such instances, it would be logical to incorporate this information into the design of test plan. Assume that upper bound $u_i(\mathbf{x})$ on each of the failure rates $\lambda_i(\mathbf{x})$ is known. Note that $u_i(\mathbf{x})$ need not be constant, it will generally be a function of covariates. Similar assumption on failure rate with constant upper bound can be seen in papers [13] and [83]. With this assumption, the Problem $\mathcal{Q}_4$ and Problem $\mathcal{Q}_5$ can be rewritten as

Problem $\mathcal{Q}_6$

$$\text{Maximize} \sum_{i=1}^{n}\left(\frac{\left(\sum_{j=1}^{k}\beta_{ij}x_j\right)^2}{r_i}\right)$$

such that

$$\sum_{i=1}^{n}\sum_{j=1}^{k}\beta_{ij}x_j = -ln(R_1)$$
$$\sum_{j=1}^{k}\beta_{ij}x_j > 0 \ \ \forall \ \ i = 1,2,...,n$$
$$\sum_{j=1}^{k}\beta_{ij}x_j \le u_i(\mathbf{x}) \ \ \forall \ \ i = 1,2,...,n$$
$$r_i \ge 1, \text{ are integers } \forall \ \ i = 1,2,...,n$$
$$\beta_{ij} \in \mathbb{R}.$$

Problem $\mathcal{Q}_7$

$$\text{Maximize} \sum_{i=1}^{n}\left(\frac{\left(\sum_{j=1}^{k}\beta_{ij}x_j\right)^2}{r_i}\right)$$

such that

$$\sum_{i=1}^{n}\sum_{j=1}^{k}\beta_{ij}x_j = -ln(R_0)$$
$$\sum_{j=1}^{k}\beta_{ij}x_j > 0 \ \ \forall \ \ i = 1,2,...,n$$
$$\sum_{j=1}^{k}\beta_{ij}x_j \le u_i(\mathbf{x}) \ \ \forall \ \ i = 1,2,...,n$$
$$r_i \ge 1, \text{ are integers } \forall \ \ i = 1,2,...,n$$
$$\beta_{ij} \in \mathbb{R}.$$

The two problems are now well behaved, since it is no longer required to simply drive one β_{ij} to an arbitrarily small value. Observe that in order for the above two problems to be feasible, it is necessary that $\sum_{i=1}^{n} u_i(\mathbf{x}) \ge -ln(R_0)$.

Define

$$\mu_{ij}^{p} = \begin{cases} \frac{u_i(\mathbf{x})}{x_j}, & \text{if } (i \neq p) \text{ and } (j = m), \\ \frac{-ln(R_0)-\sum_{i\neq p} u_i(\mathbf{x})}{x_{min}}, & \text{if } (i = p) \text{ and } (j = m), \\ 0, & \text{elsewhere.} \end{cases}$$

$$\gamma_{ij}^{t} = \begin{cases} \frac{u_i(\mathbf{x})}{x_j}, & \text{if } (i \neq t) \text{ and } (j = m), \\ \\ \frac{-ln(R_1) - \sum\limits_{i \neq p} u_i(\mathbf{x})}{x_{min}}, & \text{if } (i = t) \text{ and } (j = m), \\ \\ 0, & \text{elsewhere}, \end{cases}$$

for $p = 1, 2, ..., n$; $t = 1, 2, ..., n$; $i = 1, 2, ..., n$; $j = 1, 2, ..., k$, where x_{min} = minimum of $\{x_1, x_2, ..., x_k\}$ and m is the value of j corresponding to x_{min}. The following lemma paves the way for solving the optimization Problem $\mathcal{Q}_6$ and $\mathcal{Q}_7$.

Lemma 3.3.2. *Let $f : \mathbb{R}^{nk} \longrightarrow \mathbb{R}$ be a function defined by*

$$f(\beta_{11}, \beta_{12}, ..., \beta_{1k}, ..., \beta_{n1}, \beta_{n2}, ..., \beta_{nk}) = \sum_{i=1}^{n} \left(\left(\sum_{j=1}^{k} \beta_{ij} x_j \right)^2 / r_i \right),$$

where $x_j > 0$, $r_i \geq 1$, are integers, $\sum\limits_{j=1}^{k} \beta_{ij} x_j > 0$ and $\beta_{ij} \in \mathbb{R}, i = 1, 2, ..., n$, $j = 1, 2, ..., k$. Then f is a convex function.

Proof: (see Section 3.5.2).

Then, by using Lemma 3.3.2, it can be shown that, assuming feasibility, the optimum solution to above two maximization problem will be at any one of these μ_{ij}^{p} and γ_{ij}^{t} respectively. We note that a similar solution procedure was employed by Rajgopal and Mazumdar (1997: [83]). Define

$$\lambda_i^* = \text{minimum} \left\{ \sum_{j=1}^{k} \mu_{ij}^{p} x_j, \ \sum_{j=1}^{k} \gamma_{ij}^{t} x_j \right\}, 1 \leq i \leq n.$$

Then for $p = 1, 2, ..., n$; $t = 1, 2, ..., n$, the optimization problem with constraints 3.3.10 and 3.3.11 can be written as

Problem $\mathcal{Q}_M$
$\min\limits_{r_i} MTETC = \sum\limits_{i=1}^{n} \frac{c_i r_i}{\lambda_i^*}$
such that

$$\sum_{i=1}^{n}\left(\frac{\left(\sum_{j=1}^{k}\mu_{ij}^{p}x_j\right)^2}{r_i}\right) \leq \left(\frac{-ln(d)+ln(R_1)}{Z_{1-\alpha}}\right)^2,$$

$$\sum_{i=1}^{n}\left(\frac{\left(\sum_{j=1}^{k}\gamma_{ij}^{t}x_j\right)^2}{r_i}\right) \leq \left(\frac{-ln(d)+ln(R_0)}{Z_{\beta}}\right)^2,$$

r_i's are positive integers $\forall\ i = 1, 2, ..., n,\ \beta_{ij} \in \mathbb{R}$,

where $-ln(d) \in (-ln(R_1), -ln(R_0))$. This is a non-linear integer programming problem. Next, an algorithm is presented in following section to obtain the optimal values of λ_i^* and r_i for $i = 1, 2, ..., n$.

3.3.3 An Algorithm to Solve the Problem $\mathcal{Q}_M$

Let $z_c^*, r_i^*, \lambda_{opt}^i$ and d^* denote the optimum cost, optimum number of failures of $i-th$ component, optimal values of λ_i^* and optimum value of $-ln(d)$ respectively.

1. Read n, k, x_j, c_i, R_0, R_1 and $u_i(\mathbf{x})$ for $i = 1, 2, ..., n$ and $j = 1, 2, ..., k$.

2. Set $flag = 0$ and $z_c^* = 0$.

3. Set x_{min} = minimum of $\{x_1, x_2, ..., x_k\}$.

4. Set $minj$ = The value of j corresponding to x_{min}.

5. If $\left(\sum_{i=1}^{n} u_i(\mathbf{x}) > -lnR_0\right)$, then go to Step 6. Else feasibility condition violated, go to Step 1.

6. Set $p = 1$

7. If $p < n$ go to Step 8. Else go to Step 19.

8. Set $t = 1$.

9. If $t < n$ go to Step 10. Else go to Step 18.

10. Set λ_i^* = minimum $\left\{\sum_{j=1}^{k}\mu_{ij}^{p}x_j,\ \sum_{j=1}^{k}\gamma_{ij}^{t}x_j\right\}$.

11. Solve the optimization problem given by
$\min_{r_i} MTETC = \sum_{i=1}^{n} \frac{c_i r_i}{\lambda_i^*}$
such that
$\sum_{i=1}^{n}\left(\frac{\left(\sum_{j=1}^{k}\mu_{ij}^{p}x_j\right)^2}{r_i}\right) \leq \left(\frac{-ln(d)+ln(R_1)}{Z_{1-\alpha}}\right)^2,$
$\sum_{i=1}^{n}\left(\frac{\left(\sum_{j=1}^{k}\gamma_{ij}^{t}x_j\right)^2}{r_i}\right) \leq \left(\frac{-ln(d)+ln(R_0)}{Z_{\beta}}\right)^2,$
$r_i \geq 1,$ are integers $\forall\ i = 1, 2, ..., n.$

12. If $\sum_{i=1}^{n}\sum_{j=1}^{k}\gamma_{ij}^{t}x_j + ln(d) < \left(\frac{r_t}{\sum_{j=1}^{k}\gamma_{ij}^{t}x_j}\right)\left(\sum_{i=1}^{n}\left(\frac{\left(\sum_{j=1}^{k}\gamma_{ij}^{t}x_j\right)^2}{r_i}\right)\right)$, then go to Step 13. Else go to Step 14.

13. Set $flag = 1$.

14. If $(flag = 0)$ and $(-z > -z_c^*)$ go to Step 15. Else go to Step 16.

15. Replace $z_c^* = z, r_i^* = r_i, \lambda_{opt}^{i} = \lambda_i^*\ \forall\ i = 1, 2, ..., n$ and $d^* = -ln(d)$.

16. Set $flag = 0$.

17. Set $t = t + 1$ and go to Step 9.

18. Set $p = p + 1$ and go to Step 7.

19. Exit.

3.3.4 Analysis and Comparison of Numerical Results

In this section, the Problem $\mathcal{Q}_M$ using the algorithm designed in Section 3.3.3 is solved. Further, the results obtained by performing simulation, and sensitivity analysis are analyzed. Let $\mathbf{x} \in \mathbb{R}^3$, r_1^{**}, z_1^{**} respectively denote common sample size and optimal cost obtained by Rajgopal and Mazumdar (1996: [82]) and, r_2^{**}, z_2^{**} respectively denote common sample size and optimal cost obtained by Vellaisamy and Kumar (2010: [73]). Let z_M^* and d_M^* respectively denote optimal testing cost and optimal value of $-ln(d)$ for the Problem $\mathcal{Q}_M$. Consider a four component series system.

Let k, the number of covariates be equal to 3. The following Table 3.3 give optimal reliability test plans corresponding to the inputs given in Table 3.2.

Table 3.2: Inputs of numerical examples for four component series system

	Example 1	Example 2	Example 3	Example 4
α	0.0001	0.05	0.05	0.001
β	0.0001	0.05	0.05	0.01
R_0	0.85	0.85	0.83	0.823
R_1	0.86	0.86	0.86	0.862
$\mathbf{x}$	15, 27, 5	15, 27, 5	3, 5, 0.9	2.6, 35, 0.6
c_1	1	8	1	0.5
c_2	2	29	2	0.22
c_3	3.5	15	3.5	0.12
c_4	5	21	5	0.75
$u_1(\mathbf{x})$	$\frac{1}{1.41x_1}$	$\frac{1}{1.41x_1}$	0.04752	$\frac{1}{0.55(x_1+x_2)}$
$u_2(\mathbf{x})$	$\frac{1}{0.77x_2}$	$\frac{1}{0.77x_2}$	0.04852	0.04851
$u_3(\mathbf{x})$	$\frac{1}{4.1x_3}$	$\frac{1}{4.1x_3}$	0.04952	$\frac{1}{6.31(x_1+x_3)}$
$u_4(\mathbf{x})$	$\frac{1}{0.99(x_1+x_2)}$	$\frac{1}{0.99(x_1+x_2)}$	0.05052	0.05051

Let $O_i,\ i = 1, 2, 3, 4$, be optimal reliability test plans obtained using the inputs in Example 1, Example 2, Example 3, Example 4, respectively, given in Table 3.2.

Table 3.3: Optimal reliability test plans for 4-component series system (for inputs given in Table3.2)

	O_1	O_2	O_3	O_4
r_1^*	5444	953	18	160
r_2^*	2639	314	116	242
r_3^*	1525	729	90	27
r_4^*	688	214	78	139
z_M^*	583813	868781	26788.9	7698.14
d_M^*	0.156571	0.156356	0.165566	0.168692
λ_{opt}^1	0.0472813	0.0472813	0.00227114	0.0483559
λ_{opt}^2	0.0307111	0.0307111	0.0485172	0.04851
λ_{opt}^3	0.0487805	0.0487805	0.0495172	0.00112411
λ_{opt}^4	0.0183571	0.02405	0.0505173	0.05051
r_1^{**}	9929	1944	246	391
r_2^{**}	68576	13415	1573	2501

Remark: Observe that in case of Example 1, $r_1^{**} = 9929$, one has to test each of the 4 components equally 9929 times. Note that $\sum_{i=1}^{n} r_i^* = 10296$. Hence, the percentage of total number of components to be tested for failure is reduced by about 74% using the proposed plan, as compared to that in the paper [82]. Similarly, when $r_2^{**} = 68576$, one has to test each of the 4 components equally 68576 times. Also $\sum_{i=1}^{n} r_i^* = 10296$. Thus, using proposed plan, the percentage of total number of components to be tested for failure is reduced by about 96% as compared to that in the paper [73].

3.3.5 Comparison of Reliability Test Plans for the case of constant costs

In this section, a comparative study is presented by comparing the proposed test plan results with the test plans obtained in the papers [82] and [73] (with fixed costs). Consider the costs given in Table 3.2 as the cost of testing a unit, then for the same inputs given in Table 3.2, the following table of results are obtained:

Table 3.4: Optimal reliability test plans (for inputs given in Table 3.2) given in the papers [82] and [73]

	O_1	O_2	O_3	O_4
r_1^*	4452	836	112	94
r_2^*	2823	341	80	145
r_3^*	1301	630	62	200
r_4^*	1073	262	29	48
z_M^*	20016.5	31529	634	138.9
d_M^*	0.156451	0.156381	0.166428	0.171082
r_1^{**}	9929	1944	246	391
z_1^{**}	114183.5	141912	2829	621.69
r_2^{**}	68576	13415	1573	2501
z_2^{**}	788624	979295	18089.5	3976.59
Case A	82.47 %	77.78 %	77.59 %	77.66 %
Case B	97.46 %	96.78 %	96.50 %	96.51 %

In the above table, $Case\ A$ and $Case\ B$ are defined as follows:

$Case\ A$: Cost reduction by test plan based upon MLE approach as compared to the plan developed by Rajgopal and Mazumdar (1996: [82]).

$Case\ B$: Cost reduction by test plan based upon MLE approach as compared to the plan developed by Vellaisamy and Kumar (2010: [73]).

It can be noted that significant reductions in testing costs are obtained in $Case\ A$ and $Case\ B$, by using test plans based on MLE approach, where constant testing cost is considered.

3.3.6 A simulation study of reliability sampling plans

In this section, a simulation study to establish the fact that the reliability sampling plans derived in Section 3.3.2, meet the specified producer's and consumer's risks, namely, α and β, adequately well, is conducted.

Consider a 4 component system with 3 covariates. Let $R_0 = 0.85, R_1 = 0.86, \alpha = \beta = 0.05$, the covariate vector $X = (15, 27, 5)$, the cost vector $c = (8, 29, 15, 21)$ and the upper bounds for failure rates are $u_1 = \frac{1}{0.41x_1}, u_2 = \frac{1}{0.77x_2}, u_3 = \frac{1}{4.1x_3}, u_4 = \frac{1}{0.99(x_1+x_2)}$. Then by the proposed plan, the optimum values of r_i^*'s are $r_1^* = 836, r_2^* = 341, r_3^* = 630, r_4^* = 262$ and $z_M^* = 31529, d_M^* = 0.156381$.

Now to simulate the lifetime of the four components with $r_1^* = 836, r_2^* = 341, r_3^* = 630, r_4^* = 262$ respectively, assume that the failure rate of the four components are $\lambda_1(x) = 0.0472813, \lambda_2(x) = 0.0307111, \lambda_3(x) = 0.0487805, \lambda_4(x) = 0.02405$ respectively. Then the simulated lifetimes of four components are shown in the following figures.

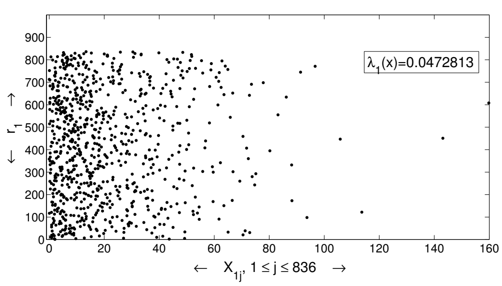

Figure 3.1: The simulated lifetime of the first component.

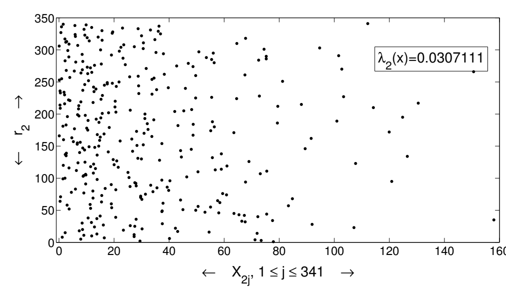

Figure 3.2: The simulated lifetime of the second component.

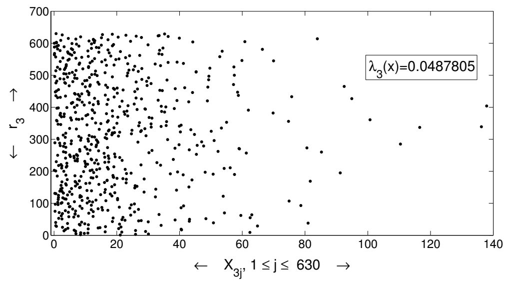

Figure 3.3: The simulated lifetime of the third component.

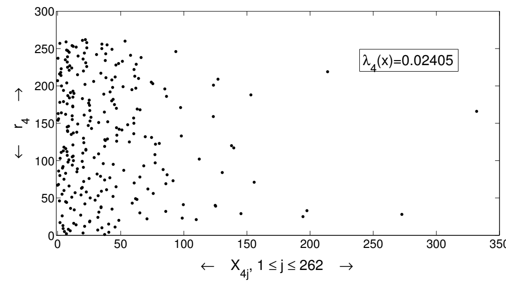

Figure 3.4: The simulated lifetime of the fourth component.

Then the total lifetimes $T_i, 1 \leq i \leq 4$ for each components are given by $T_1 = 17672.5121, T_2 = 10825.6396, T_3 = 12506.3808$ and $T_4 = 10348.2521$. Accept the system iff $\sum\limits_{i=1}^{n} \frac{r_i}{T_i} \leq -lnd$. Here $\sum\limits_{i=1}^{n} \frac{r_i}{T_i} = 0.154496967$. Clearly, $\sum\limits_{i=1}^{n} \frac{r_i}{T_i} = 0.154496967 \leq -lnd = d_M^* = 0.156381$. Hence accept the system.

Next to establish the fact that the derived sampling plan meets the specified risk α, note that $T = \sum\limits_{i=1}^{n} \frac{r_i}{T_i}$ asymptotically normally (AN) distributed random variable with mean $\sum\limits_{i=1}^{n} \lambda_i(x) = \sum\limits_{i=1}^{n} \sum\limits_{j=1}^{k} \beta_{ij} x_j$ and variance $\sum\limits_{i=1}^{n} \frac{(\lambda_i(x))^2}{r_i} = \sum\limits_{i=1}^{n} \frac{\left(\sum\limits_{j=1}^{k} \beta_{ij} x_j\right)^2}{r_i}$. Now from the inequality 3.3.3,

$$P\left(T \leq -ln(d)\right) \geq 1 - \alpha. \tag{3.3.12}$$

From the Algorithm given in Section 3.3.3, the optimum values of $\sum\limits_{j=1}^{k} \beta_{1j} x_j = 0.0472813$, $\sum\limits_{j=1}^{k} \beta_{2j} x_j = 0.0307111$, $\sum\limits_{j=1}^{k} \beta_{3j} x_j = 0.0487805$, $\sum\limits_{j=1}^{k} \beta_{4j} x_j = 0.02405$. Then $T \sim AN(0.1508229, 0.00001142465415481030)$.

The following figure shows the simulated plot of 10000 normal random variables with mean 0.1508229 and variance 0.00001142465415481030. Note that, since the variance is very small, all most all points are concentrated around the mean.

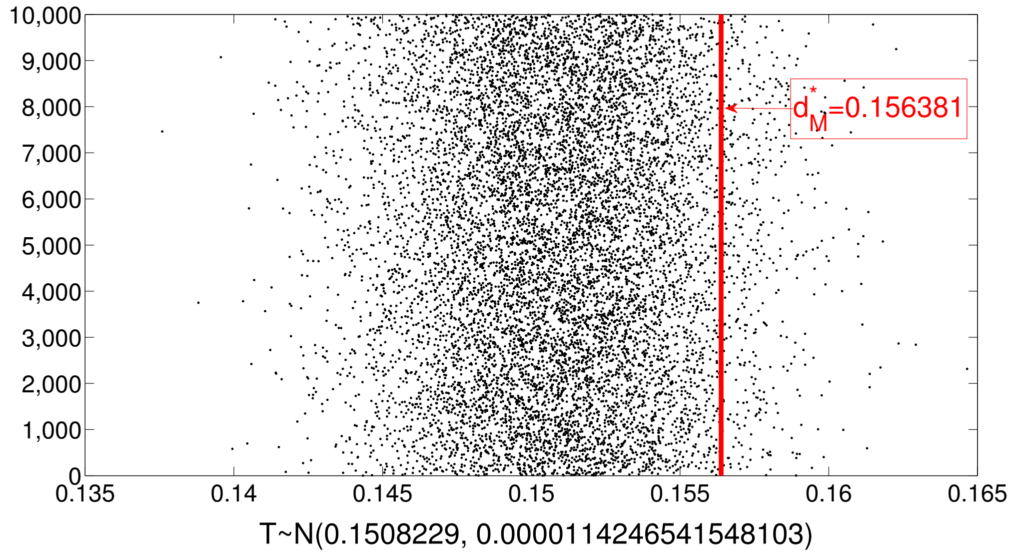

Figure 3.5: The simulated Normal random variable T corresponding to the Inequality 3.3.3.

It is observed from the simulation that $P(T \leq 0.156381) = 0.9533$, so that constraint in 3.3.12 satisfied well.

The producer's risk β plays important role in reliability acceptance test plan. A good reliability test plan should make sure that the probability of accepting a bad system is as low as possible. By simulation, it is shown that, the proposed sampling plan also meet the specified risk β. Note that $T = \sum_{i=1}^{n} \frac{r_i}{T_i}$ follows Normal distribution with mean $\sum_{i=1}^{n} \lambda_i(x) = \sum_{i=1}^{n} \sum_{j=1}^{k} \gamma_{ij} x_j$ and variance $\sum_{i=1}^{n} \frac{(\lambda_i(x))^2}{r_i} = \sum_{i=1}^{n} \frac{\left(\sum_{j=1}^{k} \gamma_{ij} x_j\right)^2}{r_i}$. Now from the Inequality 3.3.4,

$$P\left(T \leq -ln(d)\right) \leq \beta. \tag{3.3.13}$$

From the Algorithm given in Section 3.3.3, the optimum values of $\sum_{j=1}^{k} \gamma_{1j} x_j = 0.0472813$, $\sum_{j=1}^{k} \gamma_{2j} x_j = 0.0424071$, $\sum_{j=1}^{k} \gamma_{3j} x_j = 0.0487805$, $\sum_{j=1}^{k} \gamma_{4j} x_j = 0.02405$. Then $T \sim AN(0.1625189, 0.000013932544087948)$. Then by simulation, check if the Inequality 3.3.13 holds well. That is if $P(T \leq 0.156381) \leq 0.05$ holds well or not.

Using MATLAB, simulate the normal random variables T. The following figure shows the simulated plot of 10000 normal random variables with mean 0.1625189 and variance 0.000013932544087948.

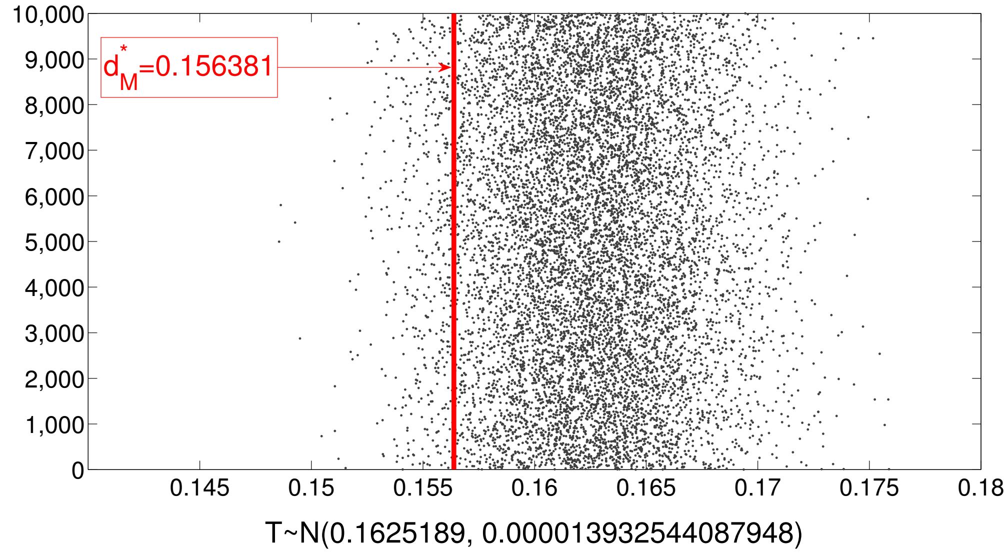

Figure 3.6: The simulated Normal random variable T corresponding to the Inequality 3.3.4.

It is observed from the simulation that $P(T \leq 0.156381) = 0.0489$, so that constraint in 3.3.13 satisfied well.

3.3.7 Sensitivity analysis

The proposed optimization model Problem $\mathcal{Q}_M$ is a non-linear integer optimization problem. Observe that the optimization Problem $\mathcal{Q}_M$ aims at minimizing the Maximum-Total-Expected-Testing-Cost (MTETC), which is not the actual cost involved in the problem. However, the actual cost will, in general be less than MTETC. The right side of the first and second constraints identified in Problem $\mathcal{Q}_M$ in Section 3.3.2, contain input parameters R_0, R_1, α, β, and c_i in the objective function. Note that the computation of λ_i^* also require the inputs R_0, R_1 and $u_i(x)$. Also the coefficients of $\frac{1}{r_i}$ in first and second constraints are calculated while running the algorithm given in Section 3.3.3, and note that they are not prefixed constants. Therefore, a detailed study on how the MTETC changes for small variations in the input parameters R_0, R_1, α, β, and c_i is presented.

As an example, consider a three component system with three covariates. Let X be the covariate vector given by (15, 27, 5). Assume that the upper bounds for failure rates are given by $u_1(x) = \frac{1}{1.2x_1}, u_2(x) = \frac{1}{0.77x_2}$ and $u_3(x) = \frac{1}{0.3x_3}$. The variation of MTETC with respect to different parameters are shown in the following figures.

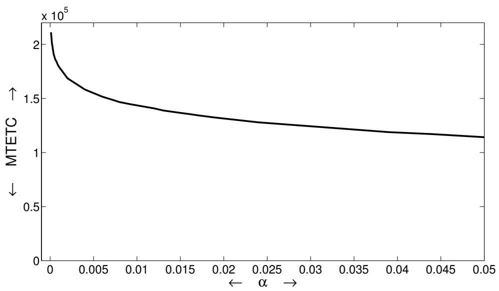

Figure 3.7: Sensitivity of MTETC to changes in α for fixed $\beta = 0.05,\ c_1 = 8,\ c_2 = 29,$ $c_3 = 15,\ R_0 = 0.86$ and $R_1 = 0.9$.

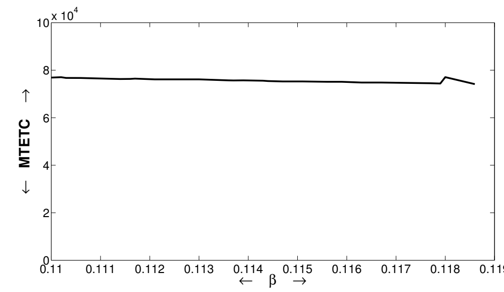

Figure 3.8: Sensitivity of MTETC to changes in β for fixed $\alpha = 0.05,\ c_1 = 8,\ c_2 = 29,$ $c_3 = 15,\ R_0 = 0.86$ and $R_1 = 0.9$.

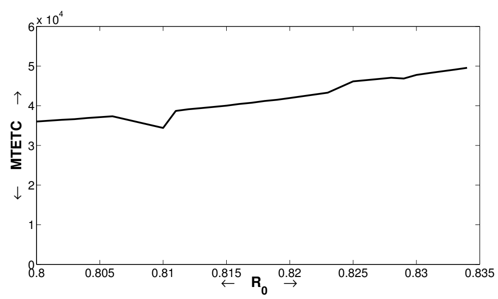

Figure 3.9: Sensitivity of MTETC to changes in R_0 for fixed $\alpha = 0.05,\ \beta = 0.1130,$ $c_1 = 8,\ c_2 = 29,\ c_3 = 15$ and $R_1 = 0.9$.

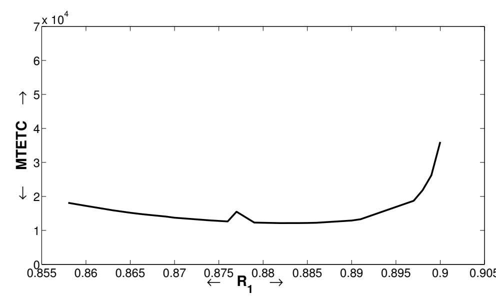

Figure 3.10: Sensitivity of MTETC to changes in R_1 for fixed $\alpha = 0.05,\ \beta = 0.1130,$ $c_1 = 8,\ c_2 = 29,\ c_3 = 15$ and $R_0 = 0.86$.

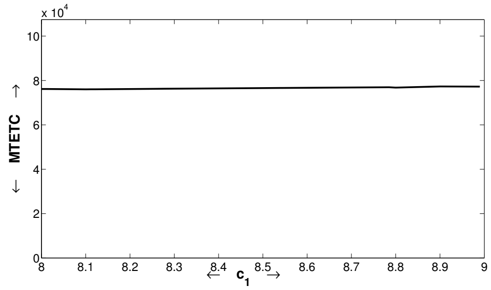

Figure 3.11: Sensitivity of MTETC to changes in c_1 for fixed $\alpha = 0.05,\ \beta = 0.05,$ $c_2 = 29,\ c_3 = 15,\ R_0 = 0.86$ and $R_1 = 0.9$.

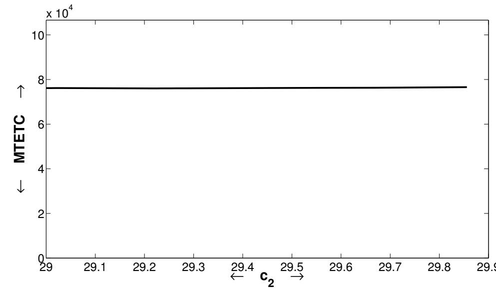

Figure 3.12: Sensitivity of MTETC to changes in c_2 for fixed $\alpha = 0.05,\ \beta = 0.05,$ $c_1 = 8,\ c_3 = 15,\ R_0 = 0.86$ and $R_1 = 0.9$.

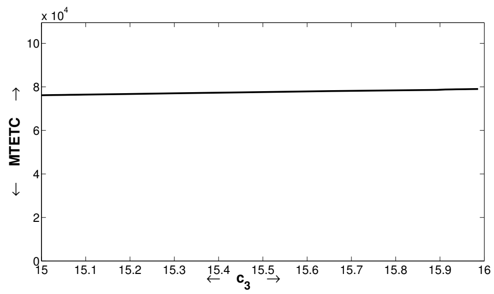

Figure 3.13: Sensitivity of MTETC to changes in c_3 for fixed $\alpha = 0.05,\ \beta = 0.05,$ $c_1 = 8,\ c_2 = 29,\ R_0 = 0.86$ and $R_1 = 0.9$.

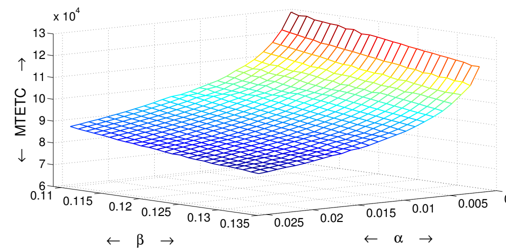

Figure 3.14: Sensitivity of MTETC to changes in α and β for fixed $c_1 = 8,\ c_2 = 29,$ $c_3 = 15,\ R_0 = 0.86$ and $R_1 = 0.9$.

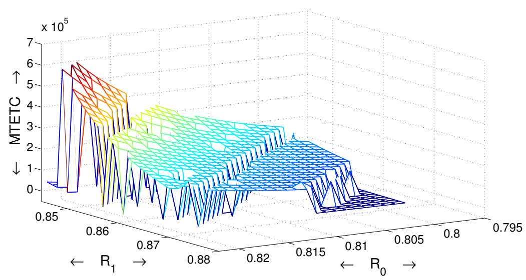

Figure 3.15: Sensitivity of MTETC to changes in R_0 and R_1 for fixed $\alpha = 0.001,$ $\beta = 0.1100,\ c_1 = 8,\ c_2 = 29,$ and $c_3 = 15$.

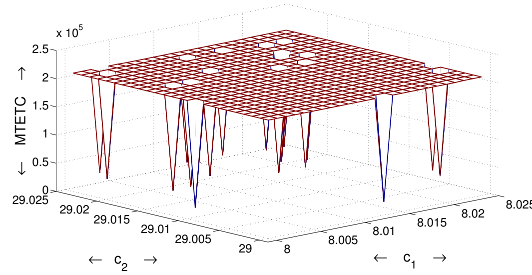

Figure 3.16: Sensitivity of MTETC to changes in c_1 and c_2 for fixed $\alpha = 0.05,\ \beta = 0.1130,$ $c_3 = 15,\ R_0 = 0.8$ and $R_1 = 0.85$.

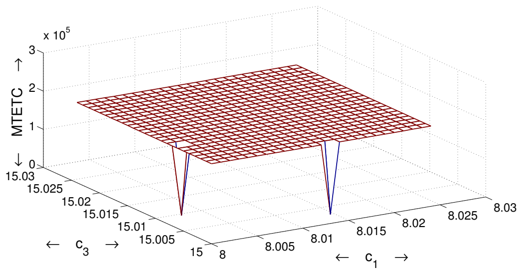

Figure 3.17: Sensitivity of MTETC to changes in c_1 and c_3 for fixed $\alpha = 0.05,\ \beta = 0.1130,\ c_2 = 29,\ R_0 = 0.8$ and $R_1 = 0.85$.

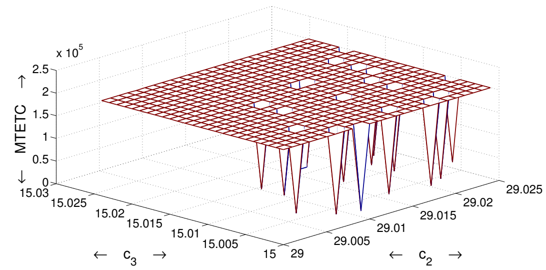

Figure 3.18: Sensitivity of MTETC to changes in c_2 and c_3 for fixed $\alpha = 0.05,\ \beta = 0.1130,\ c_1 = 8,\ R_0 = 0.8$ and $R_1 = 0.85$.

Based on numerical computations carried out to investigate the effect of variations in input parameters on MTETC, from above graphs, it is clear that the proposed model is sensitive for small variations in input parameters. However, it is observed from this sensitivity analysis study that MTETC is stable for changes in $c_1,\ c_2$ and c_3 (see Figures, 3.11, 3.12 and 3.13).

Note that the maximum-total-expected-testing-cost (MTETC) reported in Section 3.3.2 depends upon various input parameters, namely, $\alpha,\ \beta,\ R_0,\ R_1,\ c_i$. As a part of qualitative analysis of study, characterize MTETC in terms of these input parameters. Consider the producer's risk α, the changes in values of α make some variation in maximum-total-expected-testing-costs. If α is very small (that is, near zero), we observe that MTETC is increasing. If α is increasing, MTETC slowly becomes stable. Note that the values of MTETC is decreasing when α increasing (see Figure 3.7). Consider the consumer's risk β, the changes of β makes some fluctuations in MTETC, but it is almost stable. Only small changes in MTETC are made by β (see Figure 3.8).

Now consider the acceptable reliability level R_1 and unacceptable reliability level R_0. When R_1 changes, corresponding graph of MTETC is almost like a bathtub curve. For initial values of R_1, MTETC is high, then stable and then again increasing (see Figure 3.10). When R_0 changes, corresponding MTETC decreases in the interval (0.806, 0.810), and then steadily increases (see Figure 3.9).

Similarly, a small change in c_1, c_2 and c_3 are making small changes in MTETC (see

Figures 3.11, 3.12 and 3.13 respectively). Then MTETC remains almost stable for small variations in c_1, c_2 and c_3.

The variation in MTETC while changes in the values of two parameters are shown in Figures 3.14, 3.15, 3.16, 3.17 and 3.18. It can also be noted that the stability of MTETC is almost maintained when two of the parameters among $c_1,\ c_2,\ c_3$ are subjected to vary (see Figures 3.16, 3.17 and 3.18.). The same is true, when $(\alpha,\ \beta)$ are subjected to vary (see Figure 3.14).

3.4 Conclusions

In the reliability test plan proposed in the papers [82] and [73], one has to test all component equally irrespective of the testing cost. Through this work, it is shown that for a series system, the optimum design depends on the cost of the individual components and that all components need not be tested equally. Unlike most of the plans available in the literature, in the proposed plan, the acceptance constant d^* and the optimum sample size for each component depending upon the testing costs of individual components. Also, it is observed that no test plan in the literature uses prior information in the form of upper bound that is a function of covariates. However, through the proposed plan, it is observed that the use of priori information and incorporation of covariates have the advantage of savings in testing cost as illustrated through Examples. Incorporation of covariate information in modeling failure rates of components as a linear combination of covariates and, considering the testing cost as a function of time have a significant advantage in reducing the total number of components to be tested for failure. It is observed that the percentage of components to be tested for failure is reduced by about 96 %. Moreover, this type of testing the reliability of a system by obtaining data under Type-II censoring, has an advantage of obtaining realistic results, since the system is tested under normal working conditions, where the influence of risk factors are not ignored.

3.5 Appendix

3.5.1 Proof of Lemma 3.3.1

(a): Given $f(\beta_{11},\beta_{12},...,\beta_{1k},...,\beta_{n1},\beta_{n2},...,\beta_{nk}) = \frac{-ln(d)-\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{k}\beta_{ij}x_j}{\left(\sum\limits_{i=1}^{n}\left(\frac{\left(\sum\limits_{j=1}^{k}\beta_{ij}x_j\right)^2}{r_i}\right)\right)^{\frac{1}{2}}}$ and $-ln(d) \geq \sum\limits_{i=1}^{n}\sum\limits_{j=1}^{k}\beta_{ij}x_j$. Then, $\frac{\partial f}{\partial \beta_{pq}} = \frac{(-x_q)(Y_1)^{\frac{1}{2}}-(x_q)(Y_2)(Y_3)(Y_1)^{-\frac{1}{2}}}{\sum\limits_{i=1}^{n}(Y_4)}$, for all $p = 1,2,...,n;\ q = 1,2,...,k$, where, $Y_1 = \sum\limits_{i=1}^{n}\left(\frac{\left(\sum\limits_{j=1}^{k}\beta_{ij}x_j\right)^2}{r_i}\right)$, $Y_2 = -ln(d)-\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{k}\beta_{ij}x_j$, $Y_3 = \frac{\sum\limits_{j=1}^{k}\beta_{pj}x_j}{r_p}$, $Y_4 = \frac{\left(\sum\limits_{j=1}^{k}\beta_{ij}x_j\right)^2}{r_i}$.

Clearly, this is a negative quantity. Hence $\frac{\partial f}{\partial \beta_{ij}} < 0 \ \ \forall \ \ \beta_{ij}$, for $i = 1,2,...,n$ and $j = 1,2,...,k$.

(b): Given $f(\beta_{11},\beta_{12},...,\beta_{1k},...,\beta_{n1},\beta_{n2},...,\beta_{nk}) = \frac{-ln(d)-\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{k}\beta_{ij}x_j}{\left(\sum\limits_{i=1}^{n}\left(\frac{\left(\sum\limits_{j=1}^{k}\beta_{ij}x_j\right)^2}{r_i}\right)\right)^{\frac{1}{2}}}$

and $-ln(d) < \sum\limits_{i=1}^{n}\sum\limits_{j=1}^{k}\beta_{ij}x_j$. So that,

$$\frac{\partial f}{\partial \beta_{pq}} = \frac{(-x_q)(Y_1)^{\frac{1}{2}} - (x_q)(Y_2)(Y_3)(Y_1)^{-\frac{1}{2}}}{\sum\limits_{i=1}^{n}(Y_4)},$$

for all $p = 1,2,...,n;\ q = 1,2,...,k$.

If $-ln(d) < \sum_{i=1}^{n}\sum_{j=1}^{k}\beta_{ij}x_j \Rightarrow \sum_{i=1}^{n}\sum_{j=1}^{k}\beta_{ij}x_j + ln(d) > 0$, then,

$$\frac{\partial f}{\partial \beta_{pq}} = \frac{(-x_q)\left(\sum_{i=1}^{n}(Y_4)\right)^{\frac{1}{2}} + (x_q)\left(\sum_{i=1}^{n}\sum_{j=1}^{k}\beta_{ij}x_j + ln(d)\right)(Y_3)(Y_1)^{-\frac{1}{2}}}{\sum_{i=1}^{n}(Y_4)}.$$

Also, given that

$$\sum_{i=1}^{n}\sum_{j=1}^{k}\beta_{ij}x_j + ln(d) < \left(\frac{r_p}{\sum_{j=1}^{k}\beta_{pj}x_j}\right)\left(\sum_{i=1}^{n}\left(\frac{\left(\sum_{j=1}^{k}\beta_{ij}x_j\right)^2}{r_i}\right)\right),$$

then $\frac{\partial f}{\partial \beta_{pq}}$ is strictly less than the quantity

$$\frac{(-x_q)(Y_1)^{\frac{1}{2}} + (x_q)\left(\frac{r_p}{\sum_{j=1}^{k}\beta_{pj}x_j}\right)(Y_1)(Y_3)(Y_1)^{-\frac{1}{2}}}{\sum_{i=1}^{n}(Y_4)}.$$

This implies

$$\frac{\partial f}{\partial \beta_{pq}} < \frac{(-x_q)\left(\sum_{i=1}^{n}\left(\frac{\left(\sum_{j=1}^{k}\beta_{ij}x_j\right)^2}{r_i}\right)\right)^{\frac{1}{2}} + (x_q)\left(\sum_{i=1}^{n}\left(\frac{\left(\sum_{j=1}^{k}\beta_{ij}x_j\right)^2}{r_i}\right)\right)^{\frac{1}{2}}}{\sum_{i=1}^{n}\left(\frac{\left(\sum_{j=1}^{k}\beta_{ij}x_j\right)^2}{r_i}\right)}.$$

Clearly $\frac{\partial f}{\partial \beta_{pq}} < 0 \;\; \forall \;\; \beta_{pq}, p = 1,2,...,n, \; q = 1,2,...,k$. Hence, f is strictly decreasing function of $\beta_{ij} \;\; \forall \;\; i = 1,2,...,n, \;\; j = 1,2,...,k$.

3.5.2 Proof of Lemma 3.3.2

$f(\beta_{11}, \beta_{12}, ..., \beta_{1k}, ..., \beta_{n1}, \beta_{n2}, ..., \beta_{nk}) = \sum\limits_{i=1}^{n} \frac{\left(\sum\limits_{j=1}^{k} \beta_{ij} x_j\right)^2}{r_i}$, then the Hessian matrix obtained for f, is given by

$$\mathbf{H} = \begin{pmatrix} \mathbf{H_1} & \mathbf{O} & \cdots & \mathbf{O} \\ \mathbf{O} & \mathbf{H_2} & \cdots & \mathbf{O} \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{O} & \cdots & \mathbf{O} & \mathbf{H_n} \end{pmatrix},$$

where for $1 \leq i \leq n$,

$$\mathbf{H_i} = \begin{pmatrix} \frac{2x_1^2}{r_i} & \frac{2x_1x_2}{r_i} & \cdots & \frac{2x_1x_k}{r_i} \\ \frac{2x_1x_2}{r_i} & \frac{2x_2^2}{r_i} & \cdots & \frac{2x_2x_k}{r_i} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{2x_1x_k}{r_i} & \frac{2x_2x_k}{r_i} & \cdots & \frac{2x_k^2}{r_i} \end{pmatrix} \quad and \quad \mathbf{O} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Let $\mathbf{E}$ be the equivalent matrix of $\mathbf{H}$ obtained by row transformations $R_2 \longleftarrow \left(\frac{x_1}{x_2}\right) R_2 - R_1,\ R_3 \longleftarrow \left(\frac{x_1}{x_3}\right) R_3 - R_1,\ \cdots,\ R_k \longleftarrow \left(\frac{x_1}{x_k}\right) R_k - R_1$, on the matrices $\mathbf{H_1}, \mathbf{H_2}, ..., \mathbf{H_n}$ respectively. Then $\mathbf{E}$ is given by

$$\mathbf{E} = \begin{pmatrix} \mathbf{E_1} & \mathbf{O} & \cdots & \mathbf{O} \\ \mathbf{O} & \mathbf{E_2} & \cdots & \mathbf{O} \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{O} & \cdots & \mathbf{O} & \mathbf{E_n} \end{pmatrix}, \quad \mathbf{E_i} = \begin{pmatrix} \frac{2x_1^2}{r_i} & \frac{2x_1x_2}{r_i} & \cdots & \frac{2x_1x_k}{r_i} \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \quad i = 1, 2, ..., n.$$

Now we give the following two facts from linear algebra.

Fact 1: If $\mathbf{A}$ is an $n \times n$ matrix with eigenvalues $\eta_1, \eta_2, ..., \eta_n$, then $trace(\mathbf{A}) = \sum_{i=1}^{n} \eta_i$.

Fact 2: Suppose that $\mathbf{A}$ is an $n \times n$ symmetric matrix, then the rank of $\mathbf{A}$ is equal to the number of non zero eigenvalues of $\mathbf{A}$.

Since $\mathbf{H_i}$'s and $\mathbf{E_i}$'s are equivalent matrices, we have $rank(\mathbf{H_i}) = rank(\mathbf{E_i}) \ \forall \ \ i = 1, 2, ..., n$. Clearly, $rank(\mathbf{E_i}) = 1 \ \ \forall \ \ i = 1, 2, ..., n$, and hence $rank(\mathbf{H_i}) = 1 \ \ \forall \ \ i = 1, 2, ..., n$. Note that, by our assumption, $x_i > 0, \ i = 1, 2, ..., n$, and hence, elements of $\mathbf{H_i}$ are all positive, $i = 1, 2, ..., n$. Then from Fact 1 and Fact 2, and for $1 \leq i \leq n$, the matrix $\mathbf{H_i}$ has only one non-zero eigenvalue which is positive. Thus the Hessian matrix $\mathbf{H}$ corresponding to the given function is positive semi definite. Hence f is convex function.

3.5.3 Lemma 3.5.3

Let $f : R^{nk} \longrightarrow R$ be a function defined by

$$f(\eta) = \frac{-ln \ d - \sum_{i=1}^{n} \sum_{j=1}^{k} \beta_{ij} x_j}{\sqrt{\sum_{i=1}^{n} \frac{\left(\sum_{j=1}^{k} \beta_{ij} x_j \right)^2}{r_i - 2}}}$$

where $\sum_{j=1}^{k} \beta_{ij} x_j > 0, \ \ -ln \ d > 0$ for all $d \in (0, 1), \ \ x_j$'s are positive constants, r_i's are integers greater than 2 for all i, β_{ij}'s are real numbers for $i = 1, 2, ..., n$ and $j = 1, 2, ..., k$ and $\eta = (\beta_{11}, \beta_{12}, ..., \beta_{1k}, ..., \beta_{n1}, \beta_{n2}, ..., \beta_{nk})$. Then the following statements are true.

a) If $-ln \ d > \sum_{i=1}^{n} \sum_{j=1}^{k} \beta_{ij} x_j$,
then f is strictly decreasing function of β_{ij} for all $i = 1, 2, ..., n, \ \ j = 1, 2, ..., k$.

b) If $-ln\ d < \sum_{i=1}^{n}\sum_{j=1}^{k}\beta_{ij}x_j$ and

$$\sum_{i=1}^{n}\sum_{j=1}^{k}\beta_{ij}x_j + ln\ d < \left(\frac{r_i-2}{\sum_{j=1}^{k}\beta_{ij}x_j}\right)\left(\sum_{i=1}^{n}\frac{\left(\sum_{j=1}^{k}\beta_{ij}x_j\right)^2}{r_i-2}\right)$$

for all $i = 1,2,...,n$, then f is strictly decreasing function of β_{ij}.

Proof (a):

It is given that

$$f(\eta) = \frac{-ln\ d - \sum_{i=1}^{n}\sum_{j=1}^{k}\beta_{ij}x_j}{\sqrt{\sum_{i=1}^{n}\frac{\left(\sum_{j=1}^{k}\beta_{ij}x_j\right)^2}{r_i-2}}} \text{ and } \ -ln\ d > \sum_{i=1}^{n}\sum_{j=1}^{k}\beta_{ij}x_j$$

Let $Y_1 = \sum_{i=1}^{n}\frac{\left(\sum_{j=1}^{k}\beta_{ij}x_j\right)^2}{r_i-2}$, $Y_2 = -ln\ d - \sum_{i=1}^{n}\sum_{j=1}^{k}\beta_{ij}x_j$ and $Y_3 = \dfrac{\sum_{j=1}^{k}\beta_{ij}x_j}{r_i-2}$ then

$$\frac{\partial f}{\partial \beta_{ij}} = \frac{-x_j\sqrt{Y_1} - x_jY_2Y_3\left(Y_1\right)^{-\frac{1}{2}}}{Y_1}$$

Since $Y_1 > 0,\ Y_2 > 0$ (given) and $Y_3 > 0$ for all $i = 1,2,...,n;\ j = 1,2,...,k,\ \dfrac{\partial f}{\partial \beta_{ij}} < 0$ for all $\beta_{ij},\ \ i = 1,2,...,n\ \ and\ \ j = 1,2,...,k.$

Proof (b):

It is given that

$$f(\eta) = \frac{-ln\ d - \sum_{i=1}^{n}\sum_{j=1}^{k}\beta_{ij}x_j}{\sqrt{\sum_{i=1}^{n}\frac{\left(\sum_{j=1}^{k}\beta_{ij}x_j\right)^2}{r_i-2}}} \text{ and } \ -ln\ d < \sum_{i=1}^{n}\sum_{j=1}^{k}\beta_{ij}x_j$$

Let $Y_1 = \sum\limits_{i=1}^{n} \frac{\left(\sum\limits_{j=1}^{k} \beta_{ij} x_j\right)^2}{r_i - 2}$, $Y_2 = -ln\ d - \sum\limits_{i=1}^{n}\sum\limits_{j=1}^{k} \beta_{ij} x_j$ and $Y_3 = \frac{\sum\limits_{j=1}^{k} \beta_{ij} x_j}{r_i - 2}$, then

$$\frac{\partial f}{\partial \beta_{ij}} = \frac{-x_j\sqrt{Y_1} - x_j Y_2 Y_3 \left(Y_1\right)^{-\frac{1}{2}}}{Y_1}$$

If $-ln\ d < \sum\limits_{i=1}^{n}\sum\limits_{j=1}^{k} \beta_{ij} x_j$, then $\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{k} \beta_{ij} x_j + ln\ d > 0$.

That is, $-Y_2 > 0$, then

$$\frac{\partial f}{\partial \beta_{ij}} = \frac{-x_j\sqrt{Y_1} + x_j \left(-Y_2\right) Y_3 \left(Y_1\right)^{-\frac{1}{2}}}{Y_1}$$

Also given that $\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{k} \beta_{ij} x_j + ln\ d < \left(\frac{r_p - 2}{\sum\limits_{j=1}^{k} \beta_{pj} x_j}\right)\left(\sum\limits_{i=1}^{n} \frac{\left(\sum\limits_{j=1}^{k} \beta_{ij} x_j\right)^2}{r_i - 2}\right)$.

That is $-Y_2 < \left(\frac{1}{Y_3}\right) Y_1$, then we have

$$\frac{\partial f}{\partial \beta_{ij}} < \frac{-x_j\sqrt{Y_1} + x_j \left(\frac{1}{Y_3}\right) Y_1 Y_3 \left(Y_1\right)^{-\frac{1}{2}}}{Y_1}$$

This implies

$$\frac{\partial f}{\partial \beta_{ij}} < \frac{-x_j\sqrt{Y_1} + x_j \left(Y_1\right)^{\frac{1}{2}}}{Y_1} = 0$$

That is $\frac{\partial f}{\partial \beta_{ij}} < 0$ for all β_{ij}.

Hence f is strictly decreasing function of β_{ij} for all $i = 1, 2, ..., n;\ \ j = 1, 2, ..., k$.

3.5.4 Lemma 3.5.4

Let $f : R^{nk} \longrightarrow R$ be a function defined by

$$f(\eta) = \sum_{i=1}^{n} \frac{\left(\sum\limits_{j=1}^{k} \beta_{ij} x_j\right)^2}{r_i - 2},$$

where i varies from 1 to n, j varies from 1 to k, x_j's are positive constants, r_i's are integers greater than 2, β_{ij}'s are real numbers, $\sum\limits_{j=1}^{k} \beta_{ij} x_j > 0$ for all $i = 1, 2, ..., n$ and $\eta = (\beta_{11}, \beta_{12}, ..., \beta_{1k}, ..., \beta_{n1}, \beta_{n2}, ..., \beta_{nk})$. Then f is a convex function.

Proof

Here

$$f(\eta) = \sum_{i=1}^{n} \frac{\left(\sum\limits_{j=1}^{k} \beta_{ij} x_j\right)^2}{r_i - 2},$$

then the Hessian matrix is given by

$$H = \begin{pmatrix} H_1 & O & \cdots & O \\ O & H_2 & \cdots & O \\ \cdots & \cdots & \cdots & \cdots \\ O & \cdots & O & H_n \end{pmatrix}$$

where

$$H_i = \begin{pmatrix} \frac{2x_1^2}{r_i - 2} & \frac{2x_1 x_2}{r_i - 2} & \cdots & \frac{2x_1 x_k}{r_i - 2} \\ \\ \frac{2x_1 x_2}{r_i - 2} & \frac{2x_2^2}{r_i - 2} & \cdots & \frac{2x_2 x_k}{r_i - 2} \\ \\ \cdots & \cdots & \cdots & \cdots \\ \\ \frac{2x_1 x_k}{r_i - 2} & \frac{2x_2 x_k}{r_i - 2} & \cdots & \frac{2x_k^2}{r_i - 2} \end{pmatrix} \quad and \quad O = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \\ 0 & 0 & \cdots & 0 \\ \\ \cdots & \cdots & \cdots & \cdots \\ \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

where i varies from 1 to n. Let E_i be the equivalent matrix of H_i obtained by the row transformations $R_2 \longleftarrow \left(\frac{x_1}{x_2}\right) R_2 - R_1,\ R_3 \longleftarrow \left(\frac{x_1}{x_3}\right) R_3 - R_1,\ \cdots,\ R_k \longleftarrow \left(\frac{x_1}{x_k}\right) R_k - R_1$ on the matrices $H_i,\ i = 1, 2, ..., n$, then the equivalent matrix E of H is of the form

$$E = \begin{pmatrix} E_1 & O & \cdots & O \\ O & E_2 & \cdots & O \\ \cdots & \cdots & \cdots & \cdots \\ O & \cdots & O & E_n \end{pmatrix} \quad where \;\; E_i = \begin{pmatrix} \frac{2x_1^2}{r_i - 2} & \frac{2x_1 x_2}{r_i - 2} & \cdots & \frac{2x_1 x_k}{r_i - 2} \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

where i varies from 1 to n. Now we give the following two facts from linear algebra.

Fact 1: If A is an $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$, then $trace(A) = \sum_{i=1}^{n} \lambda_i$.

Fact 2: Suppose that A is an $n \times n$ symmetric matrix, then the rank of A is equal to the number of non zero eigenvalues of A.

Since H_i's and E_i's are equivalent matrices, we have $rank(H_i) = rank(E_i) \;\; \forall \;\; i = 1, 2, ..., n$. Clearly $rank(E_i) = 1 \;\; \forall \;\; i = 1, 2, ..., n$, and hence $rank(H_i) = 1 \;\; \forall \;\; i = 1, 2, ..., n$. Then from Fact 2, the number of non zero eigenvalues of H_i's is equal to one. By our assumption that $x_i > 0 \;\; \forall \;\; i$, the elements of H_i's are all positive. Now from Fact 1 it is clear that the only one non zero eigenvalue of H_i is positive $\forall \;\; i = 1, 2, ..., n$. Thus the Hessian matrix H corresponding to the given function is positive semi definite. Hence the given function is convex.

Chapter 4

Bayesian Reliability Test Plans for a Multi-component System

4.1 Introduction

When a manufacturer produces a new product, he should test the reliability of the product for a particular duration of time to make sure that the product will perform to the best as expected by the consumer. Fixing the testing time based on prior information is one of the difficult tasks. The incorporation of prior information about the failure rate in the form of upper bounds takes a significant role in determining the duration of testing time in the design of reliability test plans. Some of the references related to optimal reliability test plans for exponentially distributed lifetimes of components, with constant but unknown failure rates, are [9, 11, 14, 29, 36, 59, 83]. The optimal test times or an optimal number of components to be tested, reported under various situations in these papers are supposed to be used under normal working conditions regardless of the environment in which component testing is to be carried

[*] Some results of this chapter are published in the following paper and book chapter.

P. N. Bajeel and M. Kumar.:*Design of Optimal Bayesian Reliability Test Plans for a Series System*. International Journal of Pure and Applied Mathematics. Academic Publishers. Vol. 109 (5), pp. 125–133 (2016) - - (Scopus indexed).

M. Kumar and P.N. Bajeel.: *Introduction to System Reliability Evaluation through Bayesian Approach*. Book Chapter - Mathematical Concepts and Applications in Mechanical Engineering and Mechatronics. IGI Global - USA. pp. 130–153, (2017)- - (Scopus indexed).

out. In the paper [13], the reliability test plan for a series system is constructed by assuming a constant failure rate that depends upon the mission performed. In the paper [100], the authors have considered a parallel system of n independent components with constant failure rates, and the component testing procedure guarantees that the given consumer and producer risks are not exceeded. They assume certain restrictions on the magnitude of the unknown failure rates for guaranteeing the requirements of producer risk. The component test procedures use data from Type-I censoring and employ decision rules based on the total number of component failures during the testing periods, the number of failures obtained for each component, and the maximum likelihood estimate of the system reliability.

In this chapter, a Bayesian reliability test plan for a parallel and series systems consisting of n different components with the assumption that components have lifetimes that are exponentially distributed with non-constant failure rates $\lambda_i,\ 1 \leq i \leq n$, is considered. That is, λ_i's are assumed to be random variables having distributed as per Quasi density law function given by $g(\lambda_i) = \frac{1}{\lambda_i^k},\ k \geq 1$. Note that this is a noninformative prior used for modeling prior information for failure rates of components. Moreover, based on readily available abundant data in the industry, an upper bound for failures are also considered. This will help in solving the resulting optimization problem involved in obtaining optimal reliability test plans. In Section 4.2, optimal Bayesian reliability test plan is developed for a parallel system using noninformative Quasi-prior. This section is organized as follows: In Section 4.2.1, a likelihood function of Exponential distribution under Type-I censoring is obtained, and the posterior distribution of the failure rate is obtained in Section 4.2.2. The reliability estimate of the system for a unit time is obtained in Section 4.2.3. In Section 4.2.4, an acceptance rule based on the reliability estimate is defined. The acceptable and unacceptable reliability levels are obtained in Section 4.2.5. The Delta method for approximating the distribution of test statistic is given in Section 4.2.6. The optimal reliability test plan is constructed in Section 4.2.7, and corresponding numerical results are discussed in Section 4.2.8. In Section 4.3, optimal Bayesian reliability test plan for a series system with noninformative Quasi-prior is discussed. This Section is organized as follows: In

Section 4.3.1, likelihood function of Exponential distribution under Type-I censoring and the posterior distribution of failure rate are obtained. The reliability estimate of the system for unit time and acceptance rule based on this reliability estimate are given in Sections 4.3.2 and 4.3.3 respectively. Mean and variance of the test statistic are obtained in Section 4.3.4, and optimal test plan is developed in Section 4.3.5. The numerical results are illustrated in Section 4.3.6. Finally, conclusions are drawn in Section 4.4.

4.2 Optimal Bayesian Reliability Test Plans for a Parallel System

In this section, the problem of testing the reliability of a highly reliable parallel system with n independent components under Type-I censoring is considered. It is assumed that the lifetime of $i-th$ component follows Exponential distribution with parameter $\lambda_i,\ \lambda_i \leq u_i\ \forall\ i = 1, 2, \cdots, n$, where u_i is a known upper bound for the failure rate λ_i. Then the parallel system reliability for unit time period is given by

$$R = 1 - \prod_{i=1}^{n}(1 - e^{-\lambda_i}).$$

Since the system is highly reliable, the system reliability can be approximated as

$$R = 1 - \prod_{i=1}^{n}(1 - e^{-\lambda_i}) \simeq 1 - \prod_{i=1}^{n} \lambda_i.$$

In Bayesian paradigm, each λ_i is treated as a random variable. Now, assume that the density of λ_i follows Quasi density given by $g(\lambda_i) = \frac{1}{\lambda_i^k},\ k \geq 1$. Since the Quasi-prior with $k > 1$ gives rise to intractable optimization problem which is difficult to solve in general set up. Therefore, consider the case $g(\lambda_i) = \frac{1}{\lambda_i}$, a simple prior with $k = 1$ as the noninformative Quasi-prior for λ_i. Let $L_i,\ 1 \leq i \leq n$ be the total testing time for the $i-th$ component. Let c_i denote the cost of testing the $i-th$ component per unit time. Then the aim is to find the time periods $L_i,\ 1 \leq i \leq n$ that minimize the total testing cost subjected to Type-I and Type-II error constraints. That is, the problem is to determine the optimum values of $L_i,\ 1 \leq i \leq n$ by formulating the following optimization problem:

$$\min_{L_i} C = \sum_{i=1}^{n} c_i L_i$$

such that

$$P(\textit{Reject the system} \mid \textit{System is good}) < \alpha, \tag{4.2.1}$$

$$P(\textit{Accept the system} \mid \textit{System is bad}) \leq \beta, \tag{4.2.2}$$

where $0 < \beta < 1 - \alpha < 1$. Here, α is usually known as producer's risk, and β is known as consumer's risk.

4.2.1 Likelihood function based on Type-I censoring

Let T_{ij} denote the lifetime of $j-th$ component of type i. Since T_{ij} follows Exponential distribution with failure rate λ_i, then the probability density function of T_{ij} is given by $f(t_{ij}) = \lambda_i e^{-\lambda_i t_{ij}}$. Thus, the likelihood function based on Type-I censoring is given by

$$L(t_{ij} \mid \lambda_i) = L = \prod_{j \in \mathbb{N}} \left(\lambda_i e^{-\lambda_i t_{ij}}\right)^{\delta_j} \left(e^{-\lambda_i t_{ij}}\right)^{1-\delta_j} = \lambda_i^{X_i} e^{-\lambda_i L_i},$$

where $\delta_i = 1$ if $t_{ij} \leq L_i$, $\delta_i = 0$ if $t_{ij} > L_i$, $X_i = \sum_{j \in \mathbb{N}} \delta_j$ and $L_i = \sum_{j \in N} t_{ij}$. (See, [55]).

4.2.2 Posterior Distribution

Let $f(t_{ij}, \lambda_i)$, $g(\lambda_i \mid t_{ij})$ and $f(t_{ij} \mid \lambda_i)$ be the joint density of t_{ij} and λ_i, the conditional density of λ_i, given t_{ij}, and the conditional density of t_{ij}, given λ_i, respectively. Under the assumption that the marginal densities $m(t_{ij})$ and $g(\lambda_i)$ of t_{ij} and λ_i, respectively, satisfy the conditions required for the existence of conditional densities, the expression for $g(\lambda_i \mid t_{ij})$ and $f(t_{ij} \mid \lambda_i)$ can be written as

$$g(\lambda_i \mid t_{ij}) = \frac{f(t_{ij}, \lambda_i)}{m(t_{ij})} \text{ and } f(t_{ij} \mid \lambda_i) = \frac{f(t_{ij}, \lambda_i)}{g(\lambda_i)}.$$

Thus, the posterior distribution of λ_i is given by

$$g(\lambda_i \mid t_{ij}) = \frac{g(\lambda_i) f(t_{ij} \mid \lambda_i)}{m(t_{ij})} = \frac{g(\lambda_i) f(t_{ij} \mid \lambda_i)}{\int_{\lambda_i=0}^{\infty} g(\lambda_i) f(t_{ij} \mid \lambda_i) d\lambda_i}.$$

$$= \frac{\lambda_i^{X_i} e^{-\lambda_i L_i} \dfrac{1}{\lambda_i}}{\int_{\lambda_i=0}^{\infty} g(\lambda_i) f(t_{ij} \mid \lambda_i) d\lambda_i} = \frac{L_i^{X_i}}{\Gamma(X_i)} \lambda_i^{X_i - 1} e^{-\lambda_i L_i}. \tag{4.2.3}$$

(See, [16] for more details).

4.2.3 Reliability Estimate of the System

Note that, $E\left(\left(\lambda_i - \hat{\lambda}_i\right)^2\right) = \int \left(\lambda_i - \hat{\lambda}_i\right)^2 f(\lambda_i) d\lambda_i$. Differentiating with respect to $\hat{\lambda}_i$ and equating to zero implies, $\hat{\lambda}_i = E(\lambda_i)$. Since posterior distribution of λ_i follow Gamma distribution given by 4.2.3, it clear that $\hat{\lambda}_i = E(\lambda_i) = \dfrac{X_i}{L_i}$. (See, [53] for more details). Then an estimate of the system reliability for unit time period is obtained using the Bayesian estimator of failure rates, and which is given by

$$\hat{R} = 1 - \prod_{i=1}^{n} \hat{\lambda}_i = 1 - \prod_{i=1}^{n} \frac{X_i}{L_i}.$$

Note here that, the loss function used is squared error loss function to obtain Baye's estimate of λ_i.

4.2.4 Acceptance Rule Based on Reliability Estimate

In this section, an acceptance rule for accepting the system is defined. The proposed rule is to accept the system if the estimate of the system reliability based on Bayesian estimator of λ_i given by $\hat{R} = 1 - \prod_{i=1}^{n} \dfrac{X_i}{L_i}$ is greater than or equal to some number d, where $d \in (0, 1)$. Then, note that

$$\hat{R} \geq d \Leftrightarrow 1 - \prod_{i=1}^{n} \frac{X_i}{L_i} \geq d \Leftrightarrow \sum_{i=1}^{n} \ln\left(\frac{X_i}{L_i}\right) \leq \ln(1-d).$$

Let $\varphi = \ln(1-d)$, then the acceptance rule can be written as

$$V = \sum_{i=1}^{n} \ln\left(\frac{X_i}{L_i}\right) \leq \varphi.$$

4.2.5 Acceptable and Unacceptable Reliability Levels

A system is said to be satisfactory for unit time if R, the survival probability, is greater than or equal to R_1, the acceptable reliability level (ARL) and, it is said to be unsatisfactory if R is less than or equal to R_0, the unacceptable reliability level (URL), where R_0 and R_1 are constants such that $0 < R_0 < R_1 < 1$. Let $\varphi_1 = \ln(1 - R_1)$ and $\varphi_2 = \ln(1 - R_0)$, then the following relations are true:

$$R \geq R_1 \Leftrightarrow \prod_{i=1}^{n} \lambda_i \leq 1 - R_1 \Leftrightarrow \sum_{i=1}^{n} \ln \lambda_i \leq \varphi_1,$$

$$R \leq R_0 \Leftrightarrow \prod_{i=1}^{n} \lambda_i \geq 1 - R_0 \Leftrightarrow \sum_{i=1}^{n} \ln \lambda_i \geq \varphi_2.$$

4.2.6 Normal approximation of distribution of V using Delta method

Since the lifetime of the components is exponentially distributed, the number of failures follows Poisson distribution with mean and variance $\lambda_i L_i$. Define

$$g(X_i) = \ln\left(\frac{X_i}{L_i}\right),$$

then $g'(X_i) = \frac{1}{X_i}$. Then by Delta method $g(X_i)$ follows Normal distribution with mean $g(\mu) = \ln(\lambda_i)$ and variance

$$(g'(\mu))^2 \sigma^2(X_i) = \frac{1}{\lambda_i L_i}.$$

That is,

$$g(X_i) \sim N\left(\ln \lambda_i, \ \frac{1}{\lambda_i L_i}\right).$$

Since $V = \sum\limits_{i=1}^{n} g(X_i) = \sum\limits_{i=1}^{n} \ln\left(\frac{X_i}{L_i}\right)$, by Lindeberg central limit theorem $V = \sum\limits_{i=1}^{n} g(X_i)$ follows Normal distribution with mean $\sum\limits_{i=1}^{n} \ln(\lambda_i)$ and variance $\sum\limits_{i=1}^{n} \frac{1}{\lambda_i L_i}$. (For more details see, [74]).

4.2.7 Optimal design of the problem

Let the cost of testing the $i-th$ component per unit time be denoted by c_i. Then based on Type-I censoring scheme, the total cost of testing is $C = \sum\limits_{i=1}^{n} c_i L_i$. Then the problem is to minimize C subjected to producers risk and consumers risk. That is, the aim is to find the optimum values of L_i from the optimization problem,

$$\min_{L_i} C = \sum_{i=1}^{n} c_i L_i$$

such that

$$P(\textit{Accept the system} \mid \textit{System is good}) \geq 1 - \alpha, \tag{4.2.4}$$

$$P(\textit{Accept the system} \mid \textit{System is bad}) \leq \beta. \tag{4.2.5}$$

Using the acceptance rule defined in Section 4.2.4, the constraints 4.2.4 and 4.2.5 can be written as

$$P\left(\sum_{i=1}^{n} \ln\left(\frac{X_i}{L_i}\right) \leq \varphi \mid \sum_{i=1}^{n} \ln \lambda_i \leq \varphi_1,\ \lambda_i \leq u_i\ \forall\ i\right) \geq 1 - \alpha, \tag{4.2.6}$$

$$P\left(\sum_{i=1}^{n} \ln\left(\frac{X_i}{L_i}\right) \leq \varphi \mid \sum_{i=1}^{n} \ln \lambda_i \geq \varphi_2,\ \lambda_i \leq u_i\ \forall\ i\right) \leq \beta. \tag{4.2.7}$$

Observe that in terms of probability of acceptance, constraint 4.2.6 states that the probability of acceptance should be at least $1-\alpha$ for all combinations of λ_i values that satisfy the conditions $\sum_{i=1}^{n} \ln \lambda_i \leq \varphi_1,\ \lambda_i \leq u_i\ \forall\ i = 1, 2, \cdots, n$. That is, the minimum probability of acceptance over all such λ_i should exceed $1 - \alpha$. The constraint 4.2.7 states that the probability of acceptance should be at most β for all combinations of λ_i values that satisfy the conditions $\sum_{i=1}^{n} \ln \lambda_i \geq \varphi_2,\ \lambda_i \leq u_i\ \forall\ i = 1, 2, \cdots, n$. That is, the maximum probability of acceptance over all such λ_i should not exceed β. Therefore constraints 4.2.6 and 4.2.7 can be rewritten as

$$\min_{\lambda_i} P\left(\sum_{i=1}^{n} \ln\left(\frac{X_i}{L_i}\right) \leq \varphi \mid \sum_{i=1}^{n} \ln \lambda_i \leq \varphi_1,\ \lambda_i \leq u_i\ \forall\ i\right) \geq 1 - \alpha, \tag{4.2.8}$$

$$\max_{\lambda_i} P\left(\sum_{i=1}^{n} \ln\left(\frac{X_i}{L_i}\right) \leq \varphi \mid \sum_{i=1}^{n} \ln \lambda_i \geq \varphi_2,\ \lambda_i \leq u_i\ \forall\ i\right) \leq \beta. \tag{4.2.9}$$

The exact distribution of $\sum_{i=1}^{n} \ln\left(\frac{X_i}{L_i}\right)$ is not easy to obtain, and in order to obtain the tractable problem, approximate the distribution of $\sum_{i=1}^{n} \ln\left(\frac{X_i}{L_i}\right)$ with Normal distribution. Recall that $\sum_{i=1}^{n} \ln\left(\frac{X_i}{L_i}\right)$ asymptotically normally distributed with mean $\sum_{i=1}^{n} \ln(\lambda_i)$ and variance $\sum_{i=1}^{n} \frac{1}{\lambda_i L_i}$. Then constraints 4.2.8 and 4.2.9 can be written as

$$\min_{\lambda_i} P\left(Z \leq \frac{\varphi - \sum_{i=1}^{n} \ln(\lambda_i)}{\sqrt{\sum_{i=1}^{n} \frac{1}{\lambda_i L_i}}} \mid \sum_{i=1}^{n} \ln \lambda_i \leq \varphi_1,\ \lambda_i \leq u_i \ \forall\ i \right) \geq 1 - \alpha, \tag{4.2.10}$$

$$\max_{\lambda_i} P\left(Z \leq \frac{\varphi - \sum_{i=1}^{n} \ln(\lambda_i)}{\sqrt{\sum_{i=1}^{n} \frac{1}{\lambda_i L_i}}} \mid \sum_{i=1}^{n} \ln \lambda_i \geq \varphi_2,\ \lambda_i \leq u_i \ \forall\ i \right) \leq \beta, \tag{4.2.11}$$

where, $Z = \frac{\sum_{i=1}^{n} \ln\left(\frac{X_i}{L_i}\right) - \sum_{i=1}^{n} \ln(\lambda_i)}{\sqrt{\sum_{i=1}^{n} \frac{1}{\lambda_i L_i}}}$. Since the cumulative distribution function of standard normal random variable is strictly increasing function in its arguments, the constraints 4.2.10 and 4.2.11 can be written as

$$\min_{\lambda_i} \left(\frac{\varphi - \sum_{i=1}^{n} \ln(\lambda_i)}{\sqrt{\sum_{i=1}^{n} \frac{1}{\lambda_i L_i}}} \mid \sum_{i=1}^{n} \ln \lambda_i \leq \varphi_1,\ \lambda_i \leq u_i \ \forall\ i \right) \geq Z1 - \alpha, \tag{4.2.12}$$

$$\max_{\lambda_i} \left(\frac{\varphi - \sum_{i=1}^{n} \ln(\lambda_i)}{\sqrt{\sum_{i=1}^{n} \frac{1}{\lambda_i L_i}}} \mid \sum_{i=1}^{n} \ln \lambda_i \geq \varphi_2,\ \lambda_i \leq u_i \ \forall\ i \right) \leq Z\beta. \tag{4.2.13}$$

Note that $Z_{1-\alpha}$ and Z_β are strictly positive and negative respectively for all values of $\alpha,\ \beta < 0.5$. Hence $\varphi \in (\ln(1 - R_1), \ln(1 - R_0))$. Now, consider the left hand side of the constraint 4.2.12.

Problem B_1:

$$\min_{\lambda_i} \frac{\varphi - \sum_{i=1}^{n} \ln(\lambda_i)}{\sqrt{\sum_{i=1}^{n} \frac{1}{\lambda_i L_i}}}$$

such that

$$\sum_{i=1}^{n} \ln \lambda_i \leq \varphi_1,$$

$$\lambda_i \leq u_i \ \forall \ i = 1, 2, \cdots, n.$$

Clearly, the optimum will attain when $\sum_{i=1}^{n} \ln \lambda_i = \varphi_1$, then the optimization Problem B_1 can be rewritten as
Problem B_2:

$$\min_{\lambda_i} \frac{\varphi - \varphi_1}{\sqrt{\sum_{i=1}^{n} \frac{1}{\lambda_i L_i}}}$$

such that

$$\sum_{i=1}^{n} \ln \lambda_i = \varphi_1,$$

$$\lambda_i \leq u_i \ \forall \ i = 1, 2, \cdots, n.$$

Since the numerator is a positive and independent of λ_i, to minimize the objective function of Problem B_2, it is enough to maximize the denominator of the objective function of Problem B_2. That is,
Problem B_3:

$$\max_{\lambda_i} \sum_{i=1}^{n} \frac{1}{\lambda_i L_i}$$

such that

$$\sum_{i=1}^{n} \ln \lambda_i = \varphi_1,$$

$$\lambda_i \leq u_i \ \forall \ i = 1, 2, \cdots, n.$$

This optimization Problem B_3 can be rewritten as
Problem B_4:

$$\max_{\lambda_i} \sum_{i=1}^{n} \frac{1}{e^{\ln \lambda_i} L_i}$$

such that

$$\sum_{i=1}^{n} \ln \lambda_i = \varphi_1,$$

$$\ln \lambda_i \leq \ln u_i \ \forall \ i = 1, 2, \cdots, n.$$

This is a convex programing problem in $\ln \lambda_i$, and can be easily solved as follows.

$$\text{Define } \vartheta_i = \begin{cases} \ln u_i, & if \ i \neq j \\ \varphi_1 - \sum\limits_{i \neq j} \ln u_i, & if \ i = j \end{cases} \quad for \ j = 1, 2, \cdots, n.$$

Then it is clear that by assuming feasibility, the optimum solution to above maximization Problem B_4 will be at any one of these ϑ_i's, let it be ϑ_i^*. Then the constraint 4.2.12 can be written as

$$\frac{\varphi - \varphi_1}{\sqrt{\sum_{i=1}^{n} \frac{1}{e^{\vartheta_i^*} L_i}}} \geq Z_{1-\alpha}.$$

That is,

$$\sum_{i=1}^{n} \frac{1}{e^{\vartheta_i^*} L_i} \leq \left(\frac{\varphi - \varphi_1}{Z_{1-\alpha}} \right)^2. \tag{4.2.14}$$

Similarly, proceeding as in the case of constraint 4.2.12, and defining

$$\mu_i = \begin{cases} \ln u_i, & if\ i \neq j \\ \varphi_2 - \sum\limits_{i \neq j} \ln u_i, & if\ i = j \end{cases} \quad for\ j = 1, 2, \cdots, n,$$

the constraint 4.2.13 can be rewritten as

$$\sum_{i=1}^{n} \frac{1}{e^{\mu_i^*} L_i} \leq \left(\frac{\varphi - \varphi_2}{Z_\beta} \right)^2. \tag{4.2.15}$$

Now, for $\varphi \in (\ln(1 - R_1),\ \ln(1 - R_0))$, the final optimal design can be written as follows

$$\min_{L_i} C = \sum_{i=1}^{n} c_i L_i$$

such that

$$\sum_{i=1}^{n} \frac{1}{e^{\vartheta_i^*} L_i} \leq \left(\frac{\varphi - \varphi_1}{Z_{1-\alpha}} \right)^2, \tag{4.2.16}$$

$$\sum_{i=1}^{n} \frac{1}{e^{\mu_i^*} L_i} \leq \left(\frac{\varphi - \varphi_2}{Z_\beta} \right)^2. \tag{4.2.17}$$

The above problem can be solved easily with the help of softwares like Lingo or MATLAB.

4.2.8 Numerical results and comparison

The optimal design developed in previous section is illustrated below with the help of numerical computation. Let the number of components in a parallel system be 3 and the cost vector $\mathbf{c} = (1, 1.5, 0.75)$. Then the following Table 4.1 gives the optimal test times and total testing cost for different inputs.

Table 4.1: Numerical example for a three component parallel system

	α	β	R_0	R_1	u_1	u_2	u_3	L_1	L_2	L_3	C	$-\varphi$
1	0.1	0.1	0.95	0.9999	0.2	0.12	0.15	57.95134	10.82155	25.27973	93.14347	4.477015
2	0.1	0.1	0.99	0.9999	0.2	0.12	0.15	113.1306	26.7236	46.11182	187.7999	5.881556
3	0.1	0.1	0.95	0.999	0.12	0.9	0.15	25.68597	25.63079	35.5445	90.79052	4.344261
4	0.1	0.1	0.95	0.999	0.1	0.2	0.3	23.99899	20.91438	40.4968	85.74315	4.413437
5	0.001	0.001	0.95	0.999	0.1	0.2	0.3	42.34731	36.90438	71.45846	151.2977	4.413437
6	0.001	0.001	0.95	0.9999	0.1	0.2	0.3	26.39473	25.91707	120.2076	155.426	4.589874

Now, consider a two component parallel system with cost vector denoted by $\mathbf{c} = (c_1, c_2)$, and upper bound vector of failure rate denoted by $\mathbf{u} = (u_1, u_2)$. Corresponding to the inputs $\mathbf{c} = (1, 1.5)$, $\mathbf{u} = (0.2, 0.15)$, $R_0 = 0.97$, $R_1 = 0.999$, $\alpha = 0.1$ and $\beta = 0.1$, the optimum test times are $L_1 = 155.5193$ and $L2 = 48.40575$. The total testing cost C and optimum value of φ respectively are 228.1279 units and -4.465383.

In the plan proposed in [83], the authors have obtained the component reliability test plan for a highly reliable parallel system under Type-I censoring with optimum test times 77.12, 102.82 and corresponding testing cost 1105.32 units for the data $n = 2$, $\mathbf{c} = (1, 10)$, $\mathbf{u} = (0.2, 0.15)$, $R_0 = 0.98$, $R_1 = 0.999$, $\alpha = 0.05$ and $\beta = 0.05$. For the same data the proposed Bayesian plan gives the optimum test times $L_1 = 159.3588$ units and $L_2 = 16.8623$ units. The total testing cost C is 327.9818 units and optimum value of $\varphi = -4.94004$. On Comparison of these two results, the proposed method has about 70% savings in total testing costs.

The results are generated by running Visual C++ and LINGO11 in tandem. The programming is done in Visual C++, within which LINGO11 is called whenever an optimization required.

4.3 Optimal Bayesian Reliability Test Plans for a Series System

In this section, the problem of testing the reliability of a series system with n independent components under Type-I censoring is considered, where the $i-th$ component has exponential lifetime with unknown parameter λ_i, $\lambda_i \leq u_i \ \forall \ i = 1, 2, ..., n$, and u_i is the predefined upper bound of the failure rate λ_i. Then the series system reliability for unit time period is given by $R = \prod_{i=1}^{n} e^{-\lambda_i}$. Consider the Quasi-prior $g(\lambda_i) = \dfrac{1}{\lambda_i^k}$, $k \geq 0$, a simple prior as the non-informative Quasi-prior for λ_i. Test the $i-th$ component in $(0, \ L_i]$. As soon as the component fails, it will be replaced by an identical component, so that the testing continue till the fixed time L_i, $1 \leq i \leq n$.

4.3.1 Posterior Distribution Based on Type-I Censoring

Let T_{ij}, the lifetime of $j-th$ component of type i. Since T_{ij} follows Exponential distribution with failure rate λ_i, the probability density function of T_{ij} is given by $f(t_{ij}) = \lambda_i e^{-\lambda_i t_{ij}}$. Then the likelihood function based on Type-I censoring is given by

$$L(t_{ij} \mid \lambda_i) = L = \prod_{j\in\mathbb{N}} \left(\lambda_i e^{-\lambda_i t_{ij}}\right)^{\delta_j} \left(e^{-\lambda_i t_{ij}}\right)^{1-\delta_j} = \lambda_i^{X_i} e^{-\lambda_i L_i},$$

where $\delta_i = 1$ if $t_{ij} \leq L_i$, $\delta_i = 0$ if $t_{ij} > L_i$, $X_i = \sum\limits_{j\in\mathbb{N}} \delta_j$ and $L_i = \sum\limits_{j\in N} t_{ij}$.

Let $f(t_{ij}, \lambda_i)$, $g(\lambda_i \mid t_{ij})$ and $f(t_{ij} \mid \lambda_i)$ be the joint density of t_{ij} and λ_i, the conditional density of λ_i, given t_{ij}, and the conditional density of t_{ij}, given λ_i, respectively. Under the assumption that the marginal densities $m(t_{ij})$ and $g(\lambda_i)$ of t_{ij} and λ_i, respectively, satisfy the conditions required for the existence of conditional densities. We have $g(\lambda_i \mid t_{ij}) = \dfrac{f(t_{ij}, \lambda_i)}{m(t_{ij})}$ and $f(t_{ij} \mid \lambda_i) = \dfrac{f(t_{ij}, \lambda_i)}{g(\lambda_i)}$. Thus, the posterior distribution of λ_i is given by

$$\begin{aligned} g(\lambda_i \mid t_{ij}) &= \frac{g(\lambda_i) f(t_{ij} \mid \lambda_i)}{m(t_{ij})} = \frac{g(\lambda_i) f(t_{ij} \mid \lambda_i)}{\int_{\lambda_i=0}^{\infty} g(\lambda_i) f(t_{ij} \mid \lambda_i) d\lambda_i} \\ &= \frac{\lambda_i^{X_i} e^{-\lambda_i L_i} \dfrac{1}{\lambda_i^k}}{\int_{\lambda_i=0}^{\infty} g(\lambda_i) f(t_{ij} \mid \lambda_i) d\lambda_i} = \frac{L_i^{X_i-k+1}}{\Gamma\left(X_i - k + 1\right)} \lambda_i^{X_i-k} e^{-\lambda_i L_i}. \end{aligned}$$

4.3.2 Reliability Estimate of the System

Note that, $E\left(\left(\lambda_i - \hat{\lambda}_i\right)^2\right) = \int \left(\lambda_i - \hat{\lambda}_i\right)^2 f(\lambda_i) d\lambda_i$. On differentiating with respect to $\hat{\lambda}_i$, and equating to zero implies, $\hat{\lambda}_i = E(\lambda_i)$. That is, $\hat{\lambda}_i = E(\lambda_i) = \dfrac{X_i - k + 1}{L_i}$, since posterior density of λ_i has Gamma distribution. Note that the loss function used here is squared error loss function to obtain Baye's estimate of λ_i.

An estimate of the system reliability is obtained by using the Bayesian estimator of failure rates. Thus $\hat{R} = \prod\limits_{i=1}^{n} e^{-\hat{\lambda}_i} = \prod_{i=1}^{n} e^{-\left(\frac{X_i-k+1}{L_i}\right)}$.

4.3.3 Acceptance Rule Based on Reliability Estimate $\hat{R}$

In this section, an acceptance rule required for optimal plan is defined. The proposed rule is to accept the system if the estimate of the system reliability based on Bayesian estimator of λ_i given by $\hat{R} = \prod_{i=1}^{n} e^{-\left(\frac{X_i-k+1}{L_i}\right)}$ is greater than or equal to some number d, where $d \in (0,1)$. Then, note that

$$\hat{R} \geq d \Leftrightarrow \prod_{i=1}^{n} e^{-\left(\frac{X_i-k+1}{L_i}\right)} \geq d \Leftrightarrow \sum_{i=1}^{n} \left(\frac{X_i - k + 1}{L_i}\right) \leq -ln\ d.$$

A system is said to be satisfactory for unit time if R, the survival probability, is greater than or equal to R_1, the acceptable reliability level (ARL) and, it is said to be unsatisfactory if R is less than or equal to R_0, the unacceptable reliability level (URL), where R_0 and R_1 are constants such that $0 < R_0 < R_1 < 1$. Then the following relations are true.

$$R \geq R_1 \Leftrightarrow \prod_{i=1}^{n} e^{-\lambda_i} \geq R_1 \Leftrightarrow \sum_{i=1}^{n} \lambda_1 \leq -ln\ R_1,$$

$$R \leq R_0 \Leftrightarrow \prod_{i=1}^{n} e^{-\lambda_i} \leq R_0 \Leftrightarrow \sum_{i=1}^{n} \lambda_1 \geq -ln\ R_0.$$

4.3.4 Mean and variance of the test statistic

Consider the test statistic obtained in previous section. That is, $\dfrac{X_i - k + 1}{L_i}$. Let us find the mean and variance of $\dfrac{X_i - k + 1}{L_i}$. Since the lifetime of components is exponentially distributed, the number of failures of $i-th$ component under Type-I censoring follows Poisson distribution with mean and variance $\lambda_i L_i$. Then,

$$E\left(\frac{X_i - k + 1}{L_i}\right) = \frac{\lambda_i Li - k + 1}{L_i} \Rightarrow E\left(\sum_{i=1}^{n} \frac{X_i - k + 1}{L_i}\right) = \sum_{i=1}^{n} \frac{\lambda_i Li - k + 1}{L_i},$$

$$Var\left(\frac{X_i - k + 1}{L_i}\right) = \frac{\lambda_i}{L_i} \Rightarrow Var\left(\sum_{i=1}^{n} \frac{X_i - k + 1}{L_i}\right) = \sum_{i=1}^{n} \frac{\lambda_i}{L_i}.$$

4.3.5 Optimal Design of the Problem

Let c_i denote the cost of testing the $i-th$ component per unit time. Then the aim is to find the time periods $L_i,\ 1 \leq i \leq n$ that minimize the total testing cost subjected to Type-I and Type-II error constraints. That is, the problem is to determine the optimum values of L_i by formulating the following optimization problem:

$$\text{Minimize } C = \sum_{i=1}^{n} c_i L_i$$

such that

$$P(Accept\ the\ system \mid System\ is\ good) \geq 1-\alpha, \tag{4.3.1}$$

$$P(Accept\ the\ system \mid System\ is\ bad) \leq \beta, \tag{4.3.2}$$

where $0 < \beta,\ 1-\alpha < 1$. Here, the first constraint is usually referred to as producer's risk constraint, while the second is the consumer's risk constraint. Using the acceptance rule defined in Section 4.3.3, the constraints 4.3.1 and 4.3.2 can be written as

$$\min_{\lambda_i} P\left(\sum_{i=1}^{n} \frac{X_i - k + 1}{L_i} \leq -ln\ d \mid \sum_{i=1}^{n} \lambda_i \leq -\ ln\ R_1\right) \geq 1-\alpha, \tag{4.3.3}$$

$$\max_{\lambda_i} P\left(\sum_{i=1}^{n} \frac{X_i - k + 1}{L_i} \leq -ln\ d \mid \sum_{i=1}^{n} \lambda_i \geq -\ ln\ R_0\right) \leq \beta. \tag{4.3.4}$$

The exact distribution of $\sum_{i=1}^{n} \frac{X_i - k + 1}{L_i}$ is not easy to obtain, and in order to obtain the tractable optimization problem, it is necessary to approximate the distribution of $\sum_{i=1}^{n} \frac{X_i - k + 1}{L_i}$. Recall that $\sum_{i=1}^{n} \frac{X_i - k + 1}{L_i}$ has mean $\sum_{i=1}^{n} \frac{\lambda_i L_i - k + 1}{L_i}$ and variance $\sum_{i=1}^{n} \frac{\lambda_i}{L_i}$ (see, Section 4.3.4). Now,

$$Z = \frac{\sum_{i=1}^{n} \frac{X_i - k + 1}{L_i} - \sum_{i=1}^{n} \frac{\lambda_i L_i - k + 1}{L_i}}{\sqrt{\sum_{i=1}^{n} \frac{\lambda_i}{L_i}}}.$$

Thus by using the property of cumulative distribution function of Normal distribution, the constraints 4.3.3 and 4.3.4 can be written as

$$\min_{\lambda_i}\left(\frac{-ln\ d-\sum\limits_{i=1}^{n}\dfrac{\lambda_i L_i-k+1}{L_i}}{\sqrt{\sum\limits_{i=1}^{n}\dfrac{\lambda_i}{L_i}}}\ \Big|\ \sum_{i=1}^{n}\lambda_i\le -\ ln\ R_1\right)\ge Z_{1-\alpha}, \tag{4.3.5}$$

$$\max_{\lambda_i}\left(\frac{-ln\ d-\sum\limits_{i=1}^{n}\dfrac{\lambda_i L_i-k+1}{L_i}}{\sqrt{\sum\limits_{i=1}^{n}\dfrac{\lambda_i}{L_i}}}\ \Big|\ \sum_{i=1}^{n}\lambda_i\ge -\ ln\ R_0\right)\le Z_{\beta}. \tag{4.3.6}$$

Note that, $Z_{1-\alpha}$ and Z_β are strictly positive and negative respectively for all values of $\alpha,\ \beta<0.5$. Now, consider the optimization problem in left hand side of the constraint 4.3.5. Clearly, this optimization problem will attain the optimum when $\sum\limits_{i=1}^{n}\lambda_i=-\ ln\ R_1$, then this optimization problem can be rewritten as

$$\min_{\lambda_i}\frac{-ln\ d+ln\ R_1+(k-1)\sum\limits_{i=1}^{n}\dfrac{1}{L_i}}{\sqrt{\sum\limits_{i=1}^{n}\dfrac{\lambda_i}{L_i}}}\ \text{such that}\ \sum_{i=1}^{n}\lambda_i=-\ ln\ R_1.$$

Since the numerator is a positive and independent of λ_i, to minimize the objective function, it is enough to maximize the denominator. Now, using the priori information on upper bound of failure rate, this optimization problem can be rewritten as

$$\max_{\lambda_i}\sum_{i=1}^{n}\frac{\lambda_i}{L_i}\ \text{such that}\ \sum_{i=1}^{n}\lambda_i=-\ ln\ R_1,\ \lambda_i\le u_i\ \forall\ i=1,2,...,n.$$

This is a convex optimization problem in λ_i. Define $\vartheta_i=u_i$ if $i\neq j$ and $\vartheta_i=-ln\ R_1-\sum\limits_{i\neq j}u_i$ if $i=j$ for $j=1,2,...,n$. Then it is clear that by assuming feasibility, the optimum solution to above maximization problem will be at any one of these ϑ_i's, let it be ϑ_i^*. Then the constraint 4.3.5 can be written as

$$Z_{1-\alpha}\sqrt{\sum_{i=1}^{n}\frac{\vartheta_i^*}{L_i}-(k-1)\sum_{i=1}^{n}\frac{1}{L_i}} \leq -ln\ d + ln\ R_1. \tag{4.3.7}$$

In a similar way, by defining $\mu_i = u_i$ if $i \neq j$ and $\mu_i = -ln\ R_0 - \sum\limits_{i\neq j} u_i$ if $i = j$ for $j = 1, 2, ..., n$, and by assuming feasibility, the optimum solution of the maximization problem corresponding to constraint 4.3.6 will be at any one of these μ_i's, let it be μ_i^*, and then the constraint 4.3.6 can be rewritten as

$$Z_{\beta}\sqrt{\sum_{i=1}^{n}\frac{\mu_i^*}{L_i}-(k-1)\sum_{i=1}^{n}\frac{1}{L_i}} \geq -ln\ d + ln\ R_0. \tag{4.3.8}$$

Now, for $d \in (0, 1)$, the optimal design is a convex programming problem to minimize $C = \sum\limits_{i=1}^{n} c_i L_i$ subjected to constraints 4.3.7 and 4.3.8. It can be solved easily by using softwares like Lingo or MATLAB.

4.3.6 Numerical Results for illustration of test plans

The method presented in the previous Section 4.3.5 is illustrated below with the help of numerical computation. Let the number of components in a series system be 3 and the cost vector $c = (1, 1.5, 2)$ and the upper bound vector of failure rate $u = (0.07, 0.05, 0.07)$. Then the following Table 4.2 gives the optimal test times and total testing cost for different inputs.

Table 4.2: Numerical examples for a three component series system

α	β	R_0	R_1	k	L_1	L_2	L_3	C	$-ln\ d$
0.05	0.05	0.8	0.99	0	18.5457	10.543	10.8033	55.9668	0.0100503
0.05	0.05	0.8	0.99	2	46.789	32.6688	30.1852	156.163	0.0100503
0.05	0.05	0.8	0.99	0.5	10.7909	13.5219	22.4117	75.8971	0.121366
0.08	0.05	0.85	0.98	0.5	25.6508	28.7306	48.5869	165.921	0.0952889
0.05	0.08	0.85	0.98	2	62.2929	47.3764	39.8413	231.04	0.0202027
0.05	0.05	0.85	0.99	0	25.3276	11.7293	17.9093	78.7403	0.0100503

The results are generated by running Visual C++ and LINGO11 in tandem. The programming is done in Visual C++, within which LINGO11 is called whenever an optimization required.

4.4 Conclusions

In this chapter, the designing of an optimal reliability test plan for a parallel and series system with a failure rate as a random variable having quasi-density is discussed in detail. The data are obtained through Type-I censoring scheme, and the reliability estimator is obtained by estimating Bayesian estimator of failure rates obtained under squared error loss. Some numerical examples are also computed to illustrate the Bayesian approach of testing system reliability. It is observed that the proposed plan for testing the reliability of the parallel system has about 70% savings in total testing costs, as compared to that with existing test plans (see, [83]).

Chapter 5

Optimal Design of Reliability Acceptance Sampling Plan Based on Partially Accelerated Life Test (PALT)

5.1 Introduction

In life testing, acquiring a test data at a specified normal use condition requires a long period. This problem makes life testing a difficult, time consuming and a costly procedure. Under such circumstances accelerated life tests (ALTs) or partially accelerated life tests (PALTs), which can shorten the lives of test units are used. ALT and PALT differ on the conditions at which they are applied. The test units are run only at accelerated conditions in an ALT, whereas test units are run both at accelerated and normal use conditions in a PALT.

Chernoff (1962: [23]) and Bessler et al. (1962: [87]) have coined and studied the concept of accelerated life tests. The parameter, which appears in the distribution function, is considered as a specified function of an environmental stress to which an item on test can be subjected. The problems of estimation of unknown parameter and of optimal design of the testing process in both sequential and non-sequential contexts are taken into consideration. Lifetime distribution is assumed to be exponential.

PALTs are often studied under Type-I and Type-II censoring schemes by numerous authors. For example, see [77], [25], [38], [1–3], [66, 67], [4], [49], [61].

Even though there exists a number of research work related with PALT, the optimal reliability acceptance sampling plans for a product with Weibull lifetime under PALT is not addressed, satisfying the requirements of Type-I and Type-II error constraints. The purpose of this chapter is to explore the design of acceptance sampling plan for Weibull distribution, using two different life and stress relations, namely, Arrhenius and linear stress-lifetime relationships. Here, the lifetimes of units in given lot are used to derive the desired sampling plan. This type of sampling plan has advantages over the existing sampling plans in the literature. Note that in traditional acceptance sampling plan, these features are not addressed based on data obtained from ALT or PALT. An attempt is made in this chapter to address the problem of obtaining an acceptance sampling plan for Weibull distribution. Moreover, the exact distribution of the MLE of the scale parameter of Weibull distribution is obtained for constructing optimal acceptance sampling plan.

This chapter is organized as follows: First, in Section 5.2, an acceptance sampling plan for Weibull distribution using PALT using linear life-stress relation is designed. The Sections 5.2.1, 5.2.2, 5.2.3, 5.2.4 and 5.2.5 respectively discuss about PALT procedure, distributions of lifetime under normal stress condition, distributions of lifetime under accelerated stress condition, MLE of Weibull parameter using transformed data, and the distribution of MLEs. The optimal sampling plan is developed in Section 5.2.6. Secondly, in Section 5.3, an acceptance sampling plan for Weibull distribution using PALT under Arrhenius life-stress relation is designed. The optimal sampling plan is developed in Section 5.3.1. The numerical results are discussed in Section 5.4. A comparative study and discussion on linear and Arrhenius life-stress relations used in PALT are addressed in Section 5.4.1. The conclusion of this chapter is drawn Section 5.5.

5.2 Acceptance sampling plan (ASP) for Weibull distribution using PALT: Application of linear life-stress relation

In this section, acceptance sampling plan for Weibull distribution using PALT is derived through a linear life-stress relation namely, $X = \frac{T}{\lambda}$, where X denote the lifetime of a unit under accelerated stress condition, and T is the lifetime under normal stress condition, and $\lambda > 0$ is acceleration factor.

5.2.1 PALT procedure

Consider a sample of n independent and identically distributed (iid) units from a lot. Let p be the proportion of units allocated to accelerated stress condition and $1 - p$ be the proportion of units allocated to normal stress condition. Then np is the number of randomly chosen units from n iid units to be tested under accelerated stress condition and $n(1 - p)$ is the number of units randomly chosen from n units to be tested under normal stress condition. Each unit under normal stress condition is run until the occurrence of r_1 number of failures and each unit under accelerated stress condition is run until the occurrence of r_2 number of failures.

5.2.2 Distribution of lifetime under normal stress condition

Let T be the lifetime of units under normal stress condition, following Weibull distribution with known shape parameter α and unknown scale parameter θ. Then the probability density function of T is given by

$$f_1(t, \theta, \alpha) = \frac{\alpha}{\theta} \left(\frac{t}{\theta}\right)^{\alpha-1} e^{-\left(\frac{t}{\theta}\right)^{\alpha}},\ t \geq 0,\ \alpha > 0,\ \theta > 0, \tag{5.2.1}$$

where t is the value of the random variable T.

5.2.3 Distribution of lifetime under accelerated stress condition

Let X be the lifetime of unit under accelerated stress condition with the acceleration factor λ. Assume that, at accelerated stress condition, the lifetime of an unit is

$X = \frac{T}{\lambda},\ \lambda > 0$, with cumulative distribution function given by

$$F_2(X) = P(X \leq x) = P\left(\frac{T}{\lambda} \leq x\right) = P\left(T \leq \lambda x\right) = 1 - e^{-\left(\frac{\lambda x}{\theta}\right)^{\alpha}}, \tag{5.2.2}$$

and probability density function

$$f_2(x, \alpha, \lambda, \theta) = \frac{d}{dx}\left(F_2(x)\right) = \frac{\alpha\lambda}{\theta}\left(\frac{\lambda x}{\theta}\right)^{\alpha-1} e^{-\left(\frac{\lambda x}{\theta}\right)^{\alpha}},\ \alpha > 0,\ \theta > 0,\ \lambda > 0. \tag{5.2.3}$$

From equation 5.2.2 and equation 5.2.3, it is clear that X follows Weibull distribution with shape parameter α and scale parameter $\frac{\theta}{\lambda}$.

5.2.4 An MLE of Weibull parameter using transformed data

In this section, to derive an MLE of Weibull parameter, consider the transformation $Z = T^{\alpha}$. Since the shape parameter α is constant and known in the probability density function of T, the probability density function of Z is given by

$$f_3(z, \theta, \alpha) = \frac{1}{\theta^{\alpha}} e^{-\frac{z}{\theta^{\alpha}}} = \frac{1}{\delta} e^{-\frac{z}{\delta}},\ \delta > 0, \tag{5.2.4}$$

where $\delta = \theta^{\alpha}$, and the cumulative density function is

$$F_3(z) = 1 - e^{-\frac{z}{\delta}}. \tag{5.2.5}$$

Also $X = \frac{T}{\lambda}$ implies $X^{\alpha} = \left(\frac{T}{\lambda}\right)^{\alpha} = \frac{T^{\alpha}}{\lambda^{\alpha}}$. Let $Z' = X^{\alpha}$, then the cumulative distribution function of Z' is given by

$$F_4(z') = P\left(Z' \leq z'\right) = P\left(X^{\alpha} \leq z'\right) = P\left(\frac{T^{\alpha}}{\lambda^{\alpha}} \leq z'\right) = P\left(T^{\alpha} \leq z'\lambda^{\alpha}\right) = F_3(x\lambda^{\alpha}).$$

That is,

$$F_4(z') = 1 - e^{-\frac{x\lambda^{\alpha}}{\delta}} \tag{5.2.6}$$

and the probability density function of Z' is given by

$$f_4(z', \theta, \alpha, \lambda) = \frac{\lambda^{\alpha}}{\delta} e^{-\frac{x\lambda^{\alpha}}{\delta}},\ \lambda > 0,\ \delta > 0. \tag{5.2.7}$$

That is, $Z' = X^{\alpha}$ follows Exponential distribution with parameter $\frac{\lambda^{\alpha}}{\delta}$, having mean lifetime $\frac{\delta}{\lambda^{\alpha}}$.

Let $T_j,\ j=1,2,\cdots,n(1-p)$ be the lifetime of $j-th$ unit under normal stress condition and $X_j,\ j=1,2,\cdots,np$ be the lifetime of $j-th$ unit under accelerated stress condition. Then $Z_j=T_j^{\alpha},\ j=1,2,\cdots,n(1-p)$ follows Exponential distribution with parameter δ and $Z'_j=X_j^{\alpha},\ j=1,2,\cdots,np$ follows exponential distribution with parameter $\frac{\lambda^{\alpha}}{\delta}$. Consider the likelihood function L of $Z'_j,\ j=1,2,\cdots,np$ and $Z_j,\ j=1,2,\cdots,n(1-p)$, and which is given by

$$\frac{1}{\delta}e^{\frac{-z_1}{\delta}}\frac{1}{\delta}e^{\frac{-z_2}{\delta}}\cdots\frac{1}{\delta}e^{\frac{-z_{r_1}}{\delta}}\left(e^{\frac{-z_{r_1}}{\delta}}\right)^{n(1-p)-r_1}\frac{\lambda^{\alpha}}{\delta}e^{\frac{-z'_1\lambda^{\alpha}}{\delta}}\frac{\lambda^{\alpha}}{\delta}e^{\frac{-z'_2\lambda^{\alpha}}{\delta}}\cdots$$
$$\frac{\lambda^{\alpha}}{\delta}e^{\frac{-z'_{r_2}\lambda^{\alpha}}{\delta}}\left(\frac{\lambda^{\alpha}}{\delta}e^{\frac{-z'_{r_2}\lambda^{\alpha}}{\delta}}\right)^{np-r_2}.$$

That is,

$$L=\left(\frac{1}{\delta}\right)^{r_1}\left(\frac{\lambda^{\alpha}}{\delta}\right)^{r_2}e^{-\frac{1}{\delta}\sum_{j=1}^{r_1}z_j}e^{-(n(1-p)-r_1)\frac{z_{r_1}}{\delta}}e^{-\frac{\lambda^{\alpha}}{\delta}\sum_{j=1}^{r_2}z'_j}e^{-(np-r_2)\frac{z'_{r_2}\lambda^{\alpha}}{\delta}}.$$

Then the log likelihood function, $ln(L)$ is given by

$$-r_1\,ln(\delta)+r_2\,ln(\lambda^{\alpha})-r_2\,ln(\delta)-\frac{1}{\delta}\sum_{j=1}^{r_1}z_j-(n(1-p)-r_1)\frac{z_{r_1}}{\delta}-\frac{\lambda^{\alpha}}{\delta}\sum_{j=1}^{r_2}z'_j-(np-r_2)\frac{z'_{r_2}\lambda^{\alpha}}{\delta}.$$

To find the maximum likelihood estimator of δ and λ^{α}, equate $\frac{\partial ln(L)}{\partial\delta}$ and $\frac{\partial ln(L)}{\partial\lambda}$ to zero. That is,

$$\frac{\partial ln(L)}{\partial\delta}=0\Rightarrow-\frac{r_1}{\delta}-\frac{r_2}{\delta}+\frac{\sum_{j=1}^{r_1}z_j}{\delta^2}+\frac{(n(1-p)-r_1)z_{r_1}}{\delta^2}+\frac{\lambda^{\alpha}\sum_{j=1}^{r_2}z'_j}{\delta^2}+\frac{(np-r_2)z'_{r_2}}{\delta^2}=0.$$

This implies

$$-(r_1+r_2)+\frac{\sum_{j=1}^{r_1}z_j}{\delta}+\frac{(n(1-p)-r_1)z_{r_1}}{\delta}+\frac{\lambda^{\alpha}\sum_{j=1}^{r_2}z'_j}{\delta}+\frac{(np-r_2)z'_{r_2}}{\delta}=0.$$

Then,

$$\hat{\delta}=\frac{\sum_{j=1}^{r_1}z_j+(n(1-p)-r_1)z_{r_1}+\lambda^{\alpha}\left[\sum_{j=1}^{r_2}z'_j+(np-r_2)z'_{r_2}\right]}{r_1+r_2}.$$

Let $P_1 = \sum_{j=1}^{r_1} z_j + (n(1-p) - r_1)z_{r_1}$ and $P_2 = \sum_{j=1}^{r_2} z'_j + (np - r_2)z'_{r_2}$.
Then

$$\hat{\delta} = \frac{P_1 + \lambda^\alpha P_2}{r_1 + r_2}. \tag{5.2.8}$$

Now

$$\frac{\partial ln(L)}{\partial \lambda} = 0 \Rightarrow \frac{\alpha\ r_2}{\lambda} - \frac{\alpha\ \lambda^{\alpha-1} \sum_{j=1}^{r_2} z'_j}{\delta} - \frac{(np - r_2)z'_{r_2}\alpha\lambda^{\alpha-1}}{\delta} = 0.$$

This implies

$$\frac{r_2}{\lambda} - \lambda^{\alpha-1}\left(\frac{\sum_{j=1}^{r_2} z'_j}{\delta} + \frac{(np-r_2)z'_{r_2}}{\delta}\right) = 0 \Rightarrow \frac{r_2}{\lambda} - \lambda^{\alpha-1}\left(\frac{P_2}{\delta}\right) = 0,$$

where

$$P_2 = \sum_{j=1}^{r_2} z'_j + (np - r_2)z'_{r_2}.$$

Then from equation 5.2.8,

$$\lambda^\alpha = \frac{r_2\ \delta}{P_2} = \frac{r_2\ P_1 + r_2\ \lambda^\alpha\ P_2}{P_2(r_1 + r_2)} \Rightarrow \lambda^\alpha\left(1 - \frac{r_2}{r_1 + r_2}\right) = \frac{r_2\ P_1}{P_2(r_1 + r_2)}.$$

Thus

$$\hat{\lambda}^\alpha = \frac{r_2\ P_1}{r_1\ P_2}. \tag{5.2.9}$$

Using the above equation 5.2.9, the maximum likelihood estimator of δ can be rewritten as

$$\hat{\delta} = \frac{P_1 + \lambda^\alpha P_2}{r_1 + r_2} = \frac{P_1 + \left(\frac{r_2\ P_1}{r_1\ P_2}\right) P_2}{r_1 + r_2} = \frac{P_1}{r_1}. \tag{5.2.10}$$

Since $\theta^\alpha = \delta$, note that $\hat{\theta}^\alpha = \frac{P_1}{r_1}$.

5.2.5 The distribution of $\hat{\delta}$ and $\hat{\lambda}^{\alpha}$

To find the distribution of $\hat{\delta}$, from equation 5.2.10, one can see that

$$\hat{\delta} = \frac{P_1}{r_1} = \frac{\sum_{j=1}^{r_1} z_j + (n(1-p) - r_1) z_{r_1}}{r_1} \Rightarrow \hat{\delta}^{\frac{1}{\alpha}} = \left(\frac{\sum_{j=1}^{r_1} z_j + (n(1-p) - r_1) z_{r_1}}{r_1} \right)^{\frac{1}{\alpha}}.$$

Since $\delta = \theta^{\alpha}$ and $\hat{\delta}$ is a maximum likelihood estimator of δ,

$$\hat{\theta} = \hat{\delta}^{\frac{1}{\alpha}} = \left(\frac{\sum_{j=1}^{r_1} z_j + (n(1-p) - r_1) z_{r_1}}{r_1} \right)^{\frac{1}{\alpha}}$$

is a maximum likelihood estimator of $\theta = \delta^{\frac{1}{\alpha}}$. The probability density function of the random variable $Y_1 = \hat{\delta}$ is given by Epstein and Sobel (1953: [31]), and which is given by

$$f_5(y_1) = \frac{1}{\Gamma(r_1)} \left(\frac{1}{\delta} \right)^{r_1} y_1^{r_1 - 1} e^{-\frac{r_1 \ y_1}{\delta}}, \ y_1 > 0. \tag{5.2.11}$$

Note that $\hat{\theta} = \hat{\delta}^{\frac{1}{\alpha}}$. Let $Y_2 = Y_1^{\frac{1}{\alpha}}$, then the probability density function of $\hat{\theta}$ is given by

$$f_6(y_2) = \frac{\alpha}{\Gamma(r_1)} \left(\frac{r_1}{\delta} \right)^{r_1} y_2^{r_1 \alpha - 1} e^{-\frac{r_1 \ y_2^{\alpha}}{\delta}}, \ y_2 > 0. \tag{5.2.12}$$

Also observe that

$$\hat{\lambda}^{\alpha} = \frac{r_2}{r_1} \frac{P_1}{P_2} = \frac{\left(\frac{P_1}{r_1} \right)}{\left(\frac{P_2}{r_2} \right)} = \frac{W_1}{W_2},$$

where $W_1 = \frac{P_1}{r_1}$ and $W_2 = \frac{P_2}{r_2}$. From equation 5.2.11, $\hat{\delta} = \frac{P_1}{r_1}$ follows $f_5(y_1)$ with parameter $(\delta, \ r_1)$ and $\frac{P_2}{r_2}$ follows $f_5(y_1)$ with parameter $\left(\frac{\delta}{\lambda^{\alpha}}, r_2 \right)$. Note that $\frac{2 r_1 \hat{\delta}}{\delta} \sim \chi^2(2r_1)$ and $\frac{2 r_2 \left(\frac{\hat{\delta}}{\hat{\lambda}^{\alpha}} \right)}{\frac{\delta}{\lambda^{\alpha}}} \sim \chi^2(2r_2)$ (see, [31] for proof).

To find the distribution of $\hat{\lambda}^{\alpha} = \frac{W_1}{W_2} = W$, note that the probability density function of W_1 is given by equation 5.2.11. Similarly, the probability density function of W_2

is given by

$$f_7(w_2) = \frac{1}{\Gamma(r_2)} \left(\frac{r_2\lambda^\alpha}{\delta}\right)^{r_2} w_2^{r_2-1} e^{-\frac{r_2\lambda^\alpha w_2}{\delta}}, \ w_2 > 0. \tag{5.2.13}$$

Let $f_8(w)$ be the probability density function of W and $F(w)$ be the cumulative distribution function of W. Assume that W_1 and W_2 are independent, then

$$F(w) = P\left(\frac{W_1}{W_2} \le w\right) \Rightarrow P\left(W_1 \le W_2 w\right) = F_{w_1}(w_2 w) = \int_0^\infty F_{w_1}(ww_2) f_7(w_2) dw_2,$$

where

$$\begin{aligned} F_{w_1}(ww_2) &= \int_0^{ww_2} \frac{1}{\Gamma(r_1)} \left(\frac{r_1}{\delta}\right)^{r1} w_1^{r_1-1} e^{-\frac{r_1 w_1}{\delta}} dw_1 \\ &= \frac{1}{\Gamma(r_1)} e^{-\frac{r_1 ww_2}{\delta}} \left(\frac{r_1 ww_2}{\delta}\right)^{r_1} \sum_{n=0}^{\infty} \frac{\left(\frac{r_1 ww_2}{\delta}\right)^n}{r_1(r_1+1)\cdots(r_1+n)}. \end{aligned}$$

That is,

$$F(w) = \sum_{n=0}^{\infty} \frac{\frac{1}{\Gamma(r_1)}\left(\frac{r_1 w}{\delta}\right)^{r_1} \frac{1}{\Gamma(r_2)}\left(\frac{r_2\lambda^\alpha}{\delta}\right)^{r_2}\left(\frac{r_1 w}{\delta}\right)^n}{r_1(r_1+1)\cdots(r_1+n)} \int_0^\infty e^{-\frac{r_1 w_2 w}{\delta}} w_2^{r_1+n} w_2^{r_2-1} e^{-\frac{r_2\lambda^\alpha w_2}{\delta}} dw_2.$$

This implies

$$F(w) = \sum_{n=0}^{\infty} \frac{\frac{1}{\Gamma(r_1)}\frac{1}{\Gamma(r_2)}\left(\frac{r_1 w}{\delta}\right)^{r_1+n}\left(\frac{r_2\lambda^\alpha}{\delta}\right)^{r_2}}{r_1(r_1+1)\cdots(r_1+n)} \int_0^\infty w_2^{r_1+r_2+n-1} e^{-w_2\left(\frac{r_1 w}{\delta}+\frac{r_2\lambda^\alpha}{\delta}\right)} dw_2.$$

Hence

$$F(w) = \sum_{n=0}^{\infty} \frac{\frac{1}{\Gamma(r_1)}\frac{1}{\Gamma(r_2)}\left(\frac{r_1 w}{\delta}\right)^{r_1+n}\left(\frac{r_2\lambda^\alpha}{\delta}\right)^{r_2}}{r_1(r_1+1)\cdots(r_1+n)} \frac{\Gamma(r_1+r_2+n)}{\left(\frac{r_1 w}{\delta}+\frac{r_2\lambda^\alpha}{\delta}\right)^2}. \tag{5.2.14}$$

Observe that in 5.2.14, as $w \to \infty$, $F(w) \to \infty$. That is, $F(w)$ is not a cumulative distribution function. Thus the assumption, W_1 and W_2 are independent, is wrong and hence W_1 and W_2 are dependent. Since the joint density function of W_1 and W_2 is unknown, one cannot get a closed-form expression for the probability density function of W, that is, closed-form expression for probability density function of $\hat{\lambda}^\alpha$ cannot be obtained.

5.2.6 Design of optimal sampling plan based on linear life-stress relation

Consider a lot of units having Weibull failure time with probability density function given by equation 5.2.1. In this section, a statistical testing procedure to assess whether the lifetime characteristics $\delta = \theta^{\alpha}$ adheres to the required level, is investigated. The proposed acceptance sampling plan is stated as follows:

a) Take a random sample of size n and test np units under accelerated stress condition and $n(1-p)$ units under normal stress condition.

b) Under Type-II censoring, observe r_1 failures from $n(1-p)$ units and r_2 failures from np units.

c) From observed data under Type-II censoring, calculate the MLE $\hat{\delta}$ of δ.

d) If $\hat{\delta} \geq k$, (where k is a constant to be determined satisfying the probability requirements), accept the units in the lot, otherwise reject the lot. Observe that our acceptance rule is based on the fact that, the lot will be accepted only when the mean lifetime under normal stress level exceeds some constant (that is, k).

Let δ^{*} denote the acceptable quality level (AQL) and δ^{**} denote the unacceptable quality level (UQL) of a unit in the lot. The decision on the lot as to accept or reject will be based upon the following probability requirements:

$$P(\text{Reject the lot} \mid \delta \geq \delta^{*}) \leq \alpha_1, \tag{5.2.15}$$

$$P(\text{Accept the lot} \mid \delta \leq \delta^{**}) \leq \alpha_2, \tag{5.2.16}$$

where α_1 is the producer's risk and α_2 is the consumer's risk. The unknown quantities of the test plan (p, r_1, r_2, k) are determined using an optimization problem, which minimizes the total expected testing cost (ETC) subjecting to the conditions 5.2.15 and 5.2.16. Observe that the total cost of testing involves cost associated with the testing time and cost of failed units. Since the testing time is random, to handle this situation and to derive optimal parameters of sampling plan, total expected testing cost expression is derived. Then, optimal plans are obtained minimizing total expected testing cost. Consider the following theorem which is useful in obtaining the probability of acceptance of the lot.

Theorem 5.2.1. *Let $G(t)$ be the CDF of chi- square distribution with $2r$ degrees of freedom, then $G(t)$ can be written as*

$$G(t) = 1 - e^{-t/2} \sum_{j=0}^{r-1} \frac{(t/2)^j}{j!}, \ t > 0.$$

(See, [86] for proof).

Next, define the acceptance rule for accepting the entire lot as $P_a = P(\hat{\delta} \geq k)$, where k is a constant to be determined. Using $\frac{2r_1\hat{\delta}}{\delta} \sim \chi^2(2r_1)$ and $t = \frac{2r_1 k}{\delta}$ in Theorem 5.2.1, the probability of acceptance can be obtained as

$$P_a = P\left(\frac{2r_1\hat{\delta}}{\delta} \geq k\frac{2r_1}{\delta}\right) = 1 - P\left(\frac{2r_1\hat{\delta}}{\delta} \leq k\frac{2r_1}{\delta}\right) = e^{-t/2} \sum_{j=0}^{r-1} \frac{(t/2)^j}{j!}. \qquad (5.2.17)$$

Let $S_1 = \sum\limits_{i=1}^{r_1} T_i^{\alpha} + (n(1-p) - r_1)\, T_{r_1}^{\alpha}$ be the total time of testing under normal stress condition after transformation and $S_2 = \sum\limits_{i=1}^{r_2} X_i^{\alpha} + (np - r_2) X_{r_2}^{\alpha}$ be the total time of testing under accelerated stress condition after transformation. Then, from the paper [31], one can write

$$E\left(S_1\right) = \delta \sum_{i=1}^{r_1} \frac{1}{n(1-p) - i + 1}, \qquad (5.2.18)$$

$$E\left(S_2\right) = \frac{\delta}{\lambda^{\alpha}} \sum_{i=1}^{r_2} \frac{1}{np - i + 1}. \qquad (5.2.19)$$

Hence the total expected testing time using transformed data is

$$E(S_1) + E(S_2) = \delta \left(\sum_{i=1}^{r_1} \frac{1}{n(1-p) - i + 1} + \frac{1}{\lambda^{\alpha}} \sum_{i=1}^{r_2} \frac{1}{np - i + 1} \right).$$

Let C_1 be the cost of testing a unit for unit time and C_2 be the cost of a failed unit. Then the total expected testing cost (ETC) involved in conducting the experiment is

$$ETC = \left(E(S_1) + E(S_2)\right) C_1 + \left(r_1 + r_2\right) C_2.$$

By using equations 5.2.18 and 5.2.19, ETC can be written as as

$$ETC = \delta \left(\sum_{i=1}^{r_1} \frac{1}{n(1-p)-i+1} + \frac{1}{\lambda^{\alpha}} \sum_{i=1}^{r_2} \frac{1}{np-i+1} \right) C_1 + (r_1 + r_2) C_2. \quad (5.2.20)$$

Now consider the formulation of an optimization problem which minimizes the total expected testing cost at acceptable quality level δ^*. Using inequalities 5.2.15, 5.2.16 and equations 5.2.17 and 5.2.20, the optimization problem to find (p, r_1, r_2, k) can be written as

$$\min_{p,r_1,r_2,k} \delta^* \left(\sum_{i=1}^{r_1} \frac{1}{n(1-p)-i+1} + \frac{1}{\lambda^{\alpha}} \sum_{i=1}^{r_2} \frac{1}{np-i+1} \right) C_1 + (r_1 + r_2) C_2$$

such that

$$e^{\frac{-r_1 k}{\delta}} \sum_{j=0}^{r_1-1} \frac{\left(\frac{r_1 k}{\delta}\right)^j}{j!} \geq 1 - \alpha_1, \ \delta \geq \delta^*,$$

$$e^{\frac{-r_1 k}{\delta}} \sum_{j=0}^{r_1-1} \frac{\left(\frac{r_1 k}{\delta}\right)^j}{j!} \leq \alpha_2, \ \delta \leq \delta^{**}.$$

Since δ is an unknown parameter, one can rewrite the above optimization problem as

$$\min_{p,r_1,r_2,k} \delta^* \left(\sum_{i=1}^{r_1} \frac{1}{n(1-p)-i+1} + \frac{1}{\lambda^{\alpha}} \sum_{i=1}^{r_2} \frac{1}{np-i+1} \right) C_1 + (r_1 + r_2) C_2$$

such that

$$\min_{\delta} \ e^{\frac{-r_1 k}{\delta}} \sum_{j=0}^{r_1-1} \frac{\left(\frac{r_1 k}{\delta}\right)^j}{j!} \geq 1 - \alpha_1, \ \delta \geq \delta^*, \quad (5.2.21)$$

$$\max_{\delta} \ e^{\frac{-r_1 k}{\delta}} \sum_{j=0}^{r_1-1} \frac{\left(\frac{r_1 k}{\delta}\right)^j}{j!} \leq \alpha_2, \ \delta \leq \delta^{**}. \quad (5.2.22)$$

Observe that as δ increases $e^{\frac{-r_1 k}{\delta}}$ increases. Hence the minimum with respect to δ given in Inequality 5.2.21 occurs at $\delta = \delta^*$, and the maximum with respect to δ given in Inequality 5.2.22 occurs at $\delta = \delta^{**}$. Hence the above optimization problem becomes

$$\min_{p,r_1,r_2,k} \delta^* \left(\sum_{i=1}^{r_1} \frac{1}{n(1-p)-i+1} + \frac{1}{\lambda^{\alpha}} \sum_{i=1}^{r_2} \frac{1}{np-i+1} \right) C_1 + (r_1 + r_2) C_2$$

such that

$$e^{\frac{-r_1 k}{\delta^*}} \sum_{j=0}^{r_1-1} \frac{\left(\frac{r_1 k}{\delta^*}\right)^j}{j!} \geq 1 - \alpha_1,$$

$$e^{\frac{-r_1 k}{\delta^{**}}} \sum_{j=0}^{r_1-1} \frac{\left(\frac{r_1 k}{\delta^{**}}\right)^j}{j!} \leq \alpha_2.$$

This optimization problem can be solved using genetic algorithm solver in MATLAB.

5.3 ASP using PALT: Application of Arrhenius life-stress relation

In this section, again consider a sample of n independent and identically distributed units from a lot. Let p be the proportion of units allocated for testing under accelerated stress condition and $1-p$ be the proportion of units allocated for normal stress condition. Then np is the number of randomly chosen units from n *iid* units to be tested under accelerated stress condition, and $n(1-p)$ is the number of units randomly chosen from n units for the normal stress condition. Each unit under normal stress condition is run until the occurrence of r_1 failures and each unit under accelerated stress condition is run until the occurrence of r_2 failures. Consider the Arrhenius life-stress relationship, and is given by

$$A(\zeta) = a_0 e^{\frac{a_1}{\zeta}}, \tag{5.3.1}$$

where A is a quantifiable life measure, ζ is the stress level and $a_0 > 0,\ a_1$ are the model parameter to be determined.

Let T be the lifetime of a unit under normal stress condition, having the probability density function given by Weibull distribution with parameters α and θ_1. Then the probability density function of T is given by

$$f_8(t, \theta_1, \alpha) = \frac{\alpha}{\theta_1}\left(\frac{t}{\theta_1}\right)^{\alpha-1} e^{-\left(\frac{t}{\theta_1}\right)^{\alpha}},\ t \geq 0,\ \alpha > 0,\ \theta_1 > 0. \tag{5.3.2}$$

Let X be the lifetime of a unit under accelerated stress condition and let X follow Weibull probability density function with parameters α and θ_2. Then the probability density function of X is given by

$$f_9(x, \theta_2, \alpha) = \frac{\alpha}{\theta_2}\left(\frac{x}{\theta_2}\right)^{\alpha-1} e^{-\left(\frac{x}{\theta_2}\right)^{\alpha}},\ x \geq 0,\ \alpha > 0,\ \theta_2 > 0. \tag{5.3.3}$$

As considered in Section 5.2.4, use the transformation $Z_i = T_i^{\alpha}$ and $Z_i' = X_i^{\alpha}$, then each Z_i follows $exp(\delta_1)$ and each Z_i' follows $exp(\delta_2)$, where $\delta_1 = \theta_1^{\alpha}$ and $\delta_2 = \theta_2^{\alpha}$. Let ζ_1 be the normal stress level and ζ_2 be the accelerated stress level. Under Arrhenius life-stress model, assume that

$$\theta_1^{\alpha} = a_0 e^{\frac{a_1}{\zeta_1}} \text{ and } \theta_2^{\alpha} = a_0 e^{\frac{a_1}{\zeta_2}}.$$

The likelihood function obtained from the observed data under normal stress level ζ_1 is given by

$$L_1(z_1, z_2, \cdots, z_{r_1}, \delta_1) \approx \left(\frac{1}{\delta_1}\right)^{r_1} e^{-\frac{S_1}{\delta_1}} = \left(\frac{1}{a_0 e^{\frac{a_1}{\zeta_1}}}\right)^{r_1} e^{-\frac{S_1}{\delta_1}},$$

where $S_1 = \sum_{i=1}^{r_1} z_i + (n(1-p) - r_1)\, z_{r_1}$.

The likelihood function obtained from the observed data under accelerated stress level ζ_2 is given by

$$L_2(z_1', z_2', \cdots, z_{r_2}', \delta_2) \approx \left(\frac{1}{\delta_2}\right)^{r_2} e^{-\frac{S_2}{\delta_2}} = \left(\frac{1}{a_0 e^{\frac{a_1}{\zeta_2}}}\right)^{r_2} e^{-\left(\frac{S_2}{a_0 e^{\frac{a_1}{\zeta_2}}}\right)},$$

where $S_2 = \sum_{i=1}^{r_2} z_i' + (np - r_2) z_{r_2}'$.

Now the joint likelihood function obtained using normal and accelerated stress levels is given by

$$L \approx L_1 L_2 = \left(\frac{1}{\delta_1}\right)^{r_1} e^{-\frac{S_1}{\delta_1}} \left(\frac{1}{\delta_2}\right)^{r_2} e^{-\frac{S_2}{\delta_2}}.$$

Then the log likelihood function is given by

$$\ln L = -r_1 (\ln \delta_1) - \frac{S_1}{\delta_1} - r_2 (\ln \delta_2) - \frac{S_2}{\delta_2}.$$

The normal equations, which are obtained by differentiating L partially with respect to δ_1 and δ_2 respectively are given by

$$\frac{\partial \ln L}{\partial \delta_1} = 0 \text{ and } \frac{\partial \ln L}{\partial \delta_2} = 0.$$

This implies

$$\frac{-r_1}{\delta_1} + \frac{S_1}{\delta_1^2} = 0 \text{ and } \frac{-r_2}{\delta_2} + \frac{S_2}{\delta_2^2} = 0.$$

Now MLEs of δ_1 and δ_2 are respectively given by the following equations:

$$\hat{\delta_1} = \frac{S_1}{r_1} \text{ and } \hat{\delta_2} = \frac{S_2}{r_2}. \tag{5.3.4}$$

From the equation 5.3.4, one can see that

$$\hat{a_0}e^{\frac{\hat{a_1}}{\zeta_1}} = \tfrac{S_1}{r_1} \text{ and } \hat{a_0}e^{\frac{\hat{a_1}}{\zeta_2}} = \tfrac{S_2}{r_2}.$$

By taking logarithm,

$$\ln\hat{a_0} + \frac{\hat{a_1}}{\zeta_1} = \ln\left(\frac{S_1}{r_1}\right), \tag{5.3.5}$$

$$\ln\hat{a_0} + \frac{\hat{a_1}}{\zeta_2} = \ln\left(\frac{S_2}{r_2}\right). \tag{5.3.6}$$

By subtracting equation 5.3.6 from equation 5.3.5, an estimator for a_1 is given by

$$\hat{a_1} = \frac{\zeta_1\zeta_2}{\zeta_2-\zeta_1}\left[\ln\left(\frac{S_1}{r_1}\right) - \ln\left(\frac{S_2}{r_2}\right)\right] = \frac{\zeta_1\zeta_2}{\zeta_2-\zeta_1}\ln\left(\frac{\hat{\delta}_1}{\hat{\delta}_2}\right). \tag{5.3.7}$$

By aAdding equation 5.3.5 and equation 5.3.6,

$$\hat{a_1}\left[\frac{1}{\zeta_1}+\frac{1}{\zeta_2}\right] + 2\ln\hat{a_0} = \ln\left(\frac{S_1S_2}{r_1r_2}\right).$$

That is,

$$\frac{\zeta_1\zeta_2}{\zeta_2-\zeta_1}\frac{\zeta_1+\zeta_2}{\zeta_1\zeta_2}\left[\ln\left(\frac{S_1}{r_1}\right) - \ln\left(\frac{S_2}{r_2}\right)\right] + 2\ln\hat{a_0} = \ln\left(\frac{S_1S_2}{r_1r_2}\right).$$

This implies

$$\frac{2}{\zeta_1-\zeta_2}\left[\zeta_1\ ln\left(\frac{S_1}{r_1}\right) - \zeta_2\ ln\left(\frac{S_2}{r_2}\right)\right] = 2\ln\hat{a_0}.$$

This implies

$$ln\left[\frac{\left(\frac{S_1}{r_1}\right)^{\frac{\zeta_1}{\zeta_1-\zeta_2}}}{\left(\frac{S_2}{r_2}\right)^{\frac{\zeta_2}{\zeta_1-\zeta_2}}}\right] = \ln\hat{a_0}.$$

Hence, an estimator for a_0 is given by

$$\hat{a_0} = \frac{\left(\frac{S_1}{r_1}\right)^{\frac{\zeta_1}{\zeta_1-\zeta_2}}}{\left(\frac{S_2}{r_2}\right)^{\frac{\zeta_2}{\zeta_1-\zeta_2}}} = \frac{\left(\hat{\delta}_1\right)^{\frac{\zeta_1}{\zeta_1-\zeta_2}}}{\left(\hat{\delta}_2\right)^{\frac{\zeta_2}{\zeta_1-\zeta_2}}}. \tag{5.3.8}$$

Thus $\hat{a_0}$ and $\hat{a_1}$ represent MLEs of a_0 and a_1 respectively, by using invariance property of MLE.

Let $h_0 = \frac{\zeta_1}{\zeta_1-\zeta_2}$ and $h_1 = \frac{\zeta_2}{\zeta_1-\zeta_2}$. Define $U_1 = e^{\frac{\zeta_2-\zeta_1}{\zeta_1\zeta_2}}(\hat{a_1}-a_1)$ and $U_2 = (2r_1)^{h_0}(2r_2)^{-h_1}\frac{\hat{a_0}}{a_0}$.

Using equation 5.3.7 and equation 5.3.8, U_1 and U_2 can be rewritten as

$$U_1 = \frac{\left(\frac{2S_1}{\delta_1}\right)}{\left(\frac{2S_2}{\delta_2}\right)} \left(\frac{2r_2}{2r_1}\right) \text{ and } U_2 = \frac{\left(\frac{2S_1}{\delta_1}\right)^{h_0}}{\left(\frac{2S_2}{\delta_2}\right)^{h_1}},$$

where U_1 and U_2 are pivotal quantities. The following theorem will be useful for obtaining the distributions of U_1 and U_2.

Theorem 5.3.1. *Consider S_1 and S_2 are defined in Section 5.3, then S_1 and S_2 are independent, and the distribution of $\frac{2S_i}{\delta_i} \sim \chi^2(2r_i),\ i = 1, 2$. (For proof see, [31]).*

To obtain the distributions of U_1 and U_2, the following theorem will be used.

Theorem 5.3.2. *The cumulative density functions of the pivotal quantities U_1 and U_2 are given by*

a) $F(u_1, r_1, r_2) = I_{\frac{r_1 u_1}{r_1 u_1 + r_2}}\left(\frac{r_1}{2}, \frac{r_2}{2}\right)$, where I is the regularized incomplete beta function.

b) $F(u_2, r_1.r_2) = 1 - \int\limits_0^\infty g_1(t) e^{-w/2} \sum\limits_{j=0}^{r-1} \frac{(w/2)^j}{j!} dt.$

Proof:

Let $V_1 = \frac{2S_1}{\delta_1}$ and $V_2 = \frac{2S_2}{\delta_2}$, then the following are true:

1. the cumulative distribution function of U_1 is obtained as follows.

   Since U_1 is the ratio of two independent Chi-square random variables with respective degrees of freedoms $2r_1$ and $2r_2$, $U_1 \sim F(2r_1, 2r_2)$.

2. The cumulative distribution function of U_2 is obtained as follows.

   $F_{U_2}(u) = P(U_2 \le u) = P\left(\frac{V_1^{h_0}}{V_2^{h_1}} \le u\right) = P\left(V_1^{h_0} \le u V_2^{h_1}\right) = P\left(V_1 \le (u V_2^{h_1})^{\frac{1}{h_0}}\right)$
   $= E\left(P\left(V_1 \le (u V_2^{h_1})^{\frac{1}{h_0}}\right) \Big| V_2\right) = \int\limits_0^\infty P\left(V_1 \le (u V_2^{h_1})^{\frac{1}{h_0}} \Big| V_2 = t\right) g_1(t) dt$
   $= \int\limits_0^\infty G(y) g_1(t) dt = 1 - \int\limits_0^\infty g_1(t) e^{-w/2} \sum\limits_{j=0}^{r-1} \frac{(w/2)^j}{j!} dt,$
   where $G(y)$ is the cumulative distribution function of $\chi^2(2r_1)$, $g_1(t)$ is the probability density function of $\chi^2(2r_2)$ and $w = (u V_2^{h_1})^{\frac{1}{h_0}}$.

5.3.1 Design of optimal sampling plan based on Arrhenius life-stress relation

Consider a lot of units having Weibull failure time with probability density function given in equation 5.2.1. A statistical testing procedure designed to assess whether the lifetime characteristics $\delta_1 = {\theta_1}^{\alpha}$ adheres to the required level. The proposed acceptance sampling plan is stated as follows:

a) Take a random sample of size n and test np units under accelerated stress condition and $n(1-p)$ units under normal stress condition.

b) Under Type-II censoring, observe r_1 failures from $n(1-p)$ units and r_2 failures from np units.

c) From observed data under Type-II censoring, calculate the MLE $\hat{\delta_1}$ of δ_1.

d) If $\hat{\delta_1} \geq k$, accept the units in the lot, otherwise reject the lot. Observe that our acceptance rule is based on the fact that, the lot will be accepted only when the mean lifetime under normal stress level exceeds some constant, say k, which is to be determined.

Let ${\delta_1}^*$ denote the acceptable quality level (AQL) and ${\delta_1}^{**}$ denote the unacceptable quality level (UQL) of a unit in the lot. The decision on the lot as to accept or reject will be based upon the following probability requirements:

$$P(\text{Reject the lot} \mid \delta_1 \geq {\delta_1}^*) \leq \alpha_1, \tag{5.3.9}$$

$$P(\text{Accept the lot} \mid \delta_1 \leq {\delta_1}^{**}) \leq \alpha_2, \tag{5.3.10}$$

where α_1 is the producer's risk and α_2 is the consumer's risk. The unknown quantities of the test plan (p, r_1, r_2, k) are determined by solving an optimization problem, which minimizes the total expected testing cost (ETC) subjecting to the conditions 5.2.15 and 5.2.16. The ETC that considered here is the one which is defined in Section 5.2.6.

Hence the acceptance rule for accepting the entire lot is defined as $P_a = P(\hat{\delta_1} \geq k)$, where k is a constant to be determined. Using $\frac{2r_1\hat{\delta_1}}{\delta_1} \sim \chi^2(2r_1)$ and $t = \frac{2r_1 k}{\delta_1}$ in Theorem

5.2.1, the probability of acceptance becomes,

$$P_a = P\left(\frac{2r_1\hat{\delta}_1}{\delta_1} \geq k\frac{2r_1}{\delta_1}\right) = 1 - P\left(\frac{2r_1\hat{\delta}_1}{\delta_1} \leq k\frac{2r_1}{\delta_1}\right) = e^{-t/2}\sum_{j=0}^{r-1}\frac{(t/2)^j}{j!}. \quad (5.3.11)$$

Also, note that

$$E\left(S_1\right) = \delta_1 \sum_{i=1}^{r_1} \frac{1}{n(1-p)-i+1}, \quad (5.3.12)$$

$$E\left(S_2\right) = \delta_2 \sum_{i=1}^{r_2} \frac{1}{np-i+1}. \quad (5.3.13)$$

Hence the total expected testing time using transformed data is

$$E(S_1) + E(S_2) = \delta_1 \sum_{i=1}^{r_1} \frac{1}{n(1-p)-i+1} + \delta_2 \sum_{i=1}^{r_2} \frac{1}{np-i+1}.$$

Let C_1 be the cost of testing a unit for unit time and C_2 be the cost of a failed unit, then the total expected testing cost (ETC) involved in conducting the experiment is given by

$$ETC = \left(\delta_1 \sum_{i=1}^{r_1} \frac{1}{n(1-p)-i+1} + \delta_2 \sum_{i=1}^{r_2} \frac{1}{np-i+1}\right) C_1 + \left(r_1 + r_2\right) C_2. \quad (5.3.14)$$

Since the unknowns δ_1 and δ_2 are there in the expression for ETC, we minimize ETC at δ_1^* and δ_2^* (that is, for fixed values of δ_1 and δ_2 at normal and accelerated stress level respectively). Hence using inequalities 5.3.9, 5.3.10 and equations 5.3.11 and 5.3.14, the optimization problem to find (p, r_1, r_2, k) can be written as

$$\min_{p,r_1,r_2,k}\left(\delta_1^* \sum_{i=1}^{r_1} \frac{1}{n(1-p)-i+1} + \delta_2^* \sum_{i=1}^{r_2} \frac{1}{np-i+1}\right) C_1 + \left(r_1 + r_2\right) C_2$$

such that

$$e^{\frac{-r_1 k}{\delta_1}} \sum_{j=0}^{r_1-1} \frac{\left(\frac{r_1 k}{\delta_1}\right)^j}{j!} \geq 1 - \alpha_1,\ \ \delta_1 \geq {\delta_1}^*,$$

$$e^{\frac{-r_1 k}{\delta_1}} \sum_{j=0}^{r_1-1} \frac{\left(\frac{r_1 k}{\delta_1}\right)^j}{j!} \leq \alpha_2,\ \ \delta_1 \leq {\delta_1}^{**}.$$

Since δ_1 and δ_2 are unknown parameters in above constraints, one can rewrite the above optimization problem as

$$\min_{p,r_1,r_2,k}\left(\delta_1^*\sum_{i=1}^{r_1}\frac{1}{n(1-p)-i+1}+\delta_2^*\sum_{i=1}^{r_2}\frac{1}{np-i+1}\right)C_1+(r_1+r_2)\,C_2$$

such that

$$\min_{\delta_1}\ e^{\frac{-r_1k}{\delta_1}}\sum_{j=0}^{r_1-1}\frac{\left(\frac{r_1k}{\delta_1}\right)^j}{j!}\geq 1-\alpha_1,\ \ \delta_1\geq{\delta_1}^*, \tag{5.3.15}$$

$$\max_{\delta_1}\ e^{\frac{-r_1k}{\delta_1}}\sum_{j=0}^{r_1-1}\frac{\left(\frac{r_1k}{\delta_1}\right)^j}{j!}\leq \alpha_2,\ \ \delta_1\leq{\delta_1}^{**}. \tag{5.3.16}$$

Note that as δ_1 increases $e^{\frac{-r_1k}{\delta_1}}$ increases. Hence the minimum with respect to δ_1 given in Inequality 5.3.15 occurs at $\delta_1={\delta_1}^*$ and the maximum with respect to δ_1 given in Inequality 5.3.16 occurs at $\delta_1={\delta_1}^{**}$. Hence the above optimization problem becomes

$$\min_{p,r_1,r_2,k}\left(\delta_1^*\sum_{i=1}^{r_1}\frac{1}{n(1-p)-i+1}+\delta_2^*\sum_{i=1}^{r_2}\frac{1}{np-i+1}\right)C_1+(r_1+r_2)\,C_2$$

such that

$$e^{\frac{-r_1k}{{\delta_1}^*}}\sum_{j=0}^{r_1-1}\frac{\left(\frac{r_1k}{{\delta_1}^*}\right)^j}{j!}\geq 1-\alpha_1,$$

$$e^{\frac{-r_1k}{{\delta_1}^{**}}}\sum_{j=0}^{r_1-1}\frac{\left(\frac{r_1k}{{\delta_1}^{**}}\right)^j}{j!}\leq \alpha_2.$$

This optimization problem can be solved using genetic algorithm solver in MATLAB.

5.4 Numerical results and discussions

In this section, the numerical results obtained for the optimal test plan discussed in Section 5.2 and Section 5.3 are presented. First, consider the case of acceptance sampling plan for Weibull distribution using linear life-stress model discussed in Section 5.2. Consider an example, which is included in Table 5.1 given below. For a set of values, the cost of testing a unit for unit time ($C_1=1$), the cost of a failed unit ($C_2=1$), and the value of the shape parameter of the Weibull distribution $\alpha=2$, for the choices of producer's risk $\alpha_1=0.1$, consumer's risk $\alpha_2=0.1$, acceptable quality level $\delta^*=900$, unacceptable quality level $\delta^{**}=200$, number of samples $n=40$ and the acceleration factor $\lambda=2$, the optimal values of total expected testing cost (ETC) is 1079.5, the number of failures $r_1 = 5,\ r_2 = 2$ and the lot acceptance constant $k = 329.2345$ are obtained. Thus the test plan is to accept the lot whenever $\hat{\delta}$ exceeds 329.2345, otherwise reject the lot.

Table 5.1: Optimal acceptance sampling plans using linear model

The values of $k,\ r_1$ and r_2 for various choices of $(\alpha_1,\ \alpha_2),\ (\delta^*,\ \delta^{**}),\ n,\ \lambda, \alpha = 2$ and $C_1 = 1,\ C_2 = 1$.							
$(\alpha_1,\ \alpha_2)$	$(\delta^*,\ \delta^{**})$	n	λ	p	$(r_1,\ r_2)$	k	ETC
(0.1, 0.1)	(900, 200)	40	2	0.6554	(5, 2)	329.2345	1079.5
(0.1, 0.05)	(900, 200)	40	2	0.6554	(5, 2)	428.4553	1079.5
(0.05, 0.05)	(900, 200)	40	2	0.6492	(6, 2)	348.8147	1334.3
(0.1, 0.1)	(1600, 400)	50	3	0.7301	(6, 2)	729.1776	3299.3
(0.1, 0.05)	(1600, 400)	50	3	0.7158	(9, 2)	591.6600	4625.9
(0.05, 0.05)	(1600, 400)	50	3	0.7159	(9, 2)	587.5489	4266.0
(0.05, 0.05)	(1800, 400)	30	3	0.6821	(9, 2)	587.1894	9752.2
(0.01, 0.05)	(1800, 400)	30	3	0.7240	(5, 2)	728.7510	5220.8
(0.1, 0.1)	(1800, 400)	30	3	0.7360	(4, 2)	700.3257	4010.1

Next, consider some examples in Table 5.2 to illustrate acceptance test plan using Arrhenius life-stress model. For a set of values, $C_1 = 1,\ C_2 = 1$ and $\alpha = 1$, for $\alpha_1 = 0.05,\ \alpha_2 = 0.05$, acceptable quality level $\delta_1^* = 270$, unacceptable quality level $\delta_1^{**} = 50$, $n = 50$ and the acceleration factors $\zeta_1 = 1,\ \zeta_2 = 2$, the optimal values of total expected testing cost (ETC) is 43.7748, the number of failures $r_1 = 6,\ r_2 = 1$ and the constant $k = 80.1196$ are obtained. Thus the lot will be rejected whenever $\hat{\delta}_1 > 0.801196$.

Table 5.2: Optimal acceptance sampling plans using Arrhenius model

The values of $k,\ r_1$ and r_2 for various choices of $(\alpha_1,\ \alpha_2),\ (\delta_1^*,\ \delta_1^{**}),\ n,\ \delta_2,\ \alpha = 1,\ \zeta_1 = 1,\ \zeta_2 = 2$ and $C_1 = 1,\ C_2 = 1$.							
$(\alpha_1,\ \alpha_2)$	$(\delta_1^*,\ \delta_1^{**})$	n	δ_2^*	p	$(r_1,\ r_2)$	k	ETC
(0.05, 0.05)	(270, 50)	50	37	0.1217	(6, 1)	80.1196	43.7748
(0.05, 0.05)	(504, 60)	30	42	0.0929	(5, 1)	101.6579	92.3034
(0.1, 0.1)	(1200, 160)	30	60	0.0687	(4, 1)	510.9784	153.5640
(0.05, 0.1)	(1200, 160)	30	60	0.1221	(4, 2)	362.7429	221.3770
(0.05, 0.05)	(1200, 160)	30	60	0.1221	(4, 2)	321.8990	221.3770
(0.05, 0.05)	(1500, 200)	20	67	0.1002	(5, 2)	521.5325	507.1096
(0.1, 0.05)	(1500, 200)	20	67	0.1306	(4, 2)	324.6586	309.4506
(0.1, 0.1)	(1500, 200)	20	67	0.1621	(3, 2)	277.6770	215.2409
(0.1, 0.1)	(2000, 300)	10	90	0.1	(4, 1)	402.5021	722.3354
(0.05, 0.1)	(2000, 300)	10	90	0.2	(5, 2)	403.6784	1172.7

5.4.1 Comparative study and discussion of Linear and Arrhenius life-stress models

For a set of values, the cost of testing a unit for unit time $C_1 = 1$, the cost of a failed unit $C_2 = 1$ and the value of the shape parameter of the Weibull distribution $\alpha = 2$, acceptable quality level $\delta^* = \delta_1^* = 1800$, unacceptable quality level $\delta^{**} = \delta_1^{**} = 400$, number of samples $n = 40$ and the acceleration factor $\lambda = 3$, for different values of the producer's risk α_1 and consumer's risk α_2, the following tables compare the total expected testing cost obtained under linear and Arrhenius life-stress models. Let $\%R$ denote the percentage of reduction of cost under Arrhenius model compared to that in linear model.

Table 5.3: Optimal acceptance sampling plans using linear and Arrhenius model with fixed $\alpha_1 = 0.1$

Linear model					Arrhenius model					
α_2	p	r_1, r_2	k	ETC	α_2	p	r_1, r_2	k	ETC	%R
0.1	0.7360	4,1	670.2815	4009.1	0.1	0.0687	4,1	711.891	297.2392	92.5
0.01	0.711	6,2	930.8076	6547.7	0.01	0.0494	6,1	866.5972	444.6027	93.21
0.001	0.6818	8,2	987.26	9661	0.001	0.0463	8,1	982.5218	604.2052	93.74
0.0005	0.7	8,3	1033.4	9699	0.0005	0.0333	9,1	918.0819	684.3313	92.94
0.0001	0.6	10,5	1048.5	13693	0.0001	0.0334	11,1	959.8846	771.0734	94.36

Table 5.4: Optimal acceptance sampling plans using linear and Arrhenius model with fixed $\alpha_2 = 0.1$

Linear model					Arrhenius model					
α_1	p	r_1, r_2	k	ETC	α_1	p	r_1, r_2	k	ETC	%R
0.1	0.7364	4,1	671.5725	4009.1	0.1	0.0687	4,1	698.7491	297.2392	92.5
0.01	0.6818	8,3	592.7106	9662	0.01	0.0361	8,1	598.0348	601.471	93.77
0.001	0.6291	11,3	561.2797	16247	0.001	0.0425	11,1	537.2504	865.8627	94.67
0.0005	0.6	12,3	555.6039	19408	0.0005	0.0337	12,1	542.9913	957.9362	95.06
0.0001	0.5	15,5	537.7993	42873	0.0001	0.0346	15,1	540.2972	1281.8	97.01

5.5 Conclusions

In this chapter, optimal design of acceptance sampling plans based on data obtained from partially accelerated life test are obtained by using linear and Arrhenius stress-life relationships. Type-II censoring scheme is used to obtain the data. The Maximum likelihood estimates of unknown parameter of Weibull distribution and acceleration factor are obtained for linear model. Similarly, MLEs of model parameters are obtained in case of Arrhenius model as well. Also, optimal acceptance sampling plans are developed using linear and Arrhenius stress-life relation. Several examples are presented in Table 5.1 and Table 5.2 to illustrate optimal acceptance test plans. It is observed that test cost involved in constructing acceptance sampling plan is random in nature. Hence, an expression for expected testing cost is given and the same is illustrated through several examples. However, the actual cost involved in testing may be less than that reported in this work. It is observed that when the values of producer's and consumer's risks decreases, the testing cost is increasing in plans based on both linear and Arrhenius stress-life relationships. Also, the total expected testing cost is less in plan based on Arrhenius model as compared to that in linear model, and it is observed that Arrhenius model is more cost-effective than linear model.

Chapter 6

Degradation Growth Models in the Estimation of Bayesian System Reliability

6.1 Introduction

In this chapter, as a substitute for a destructive testing procedure in estimating system reliability, readily available degradation data of systems are considered. It is difficult to estimate the system reliability for systems that are designed to achieve high reliability using data that consists of a small number of failures obtained from life tests that record only time to failure. Reliability of such systems depends on the dynamic balance between stress (which accumulate over time) and the strength. For example, a vehicle axle fails, when the depth of a crack exceeds a critical level (see, [70]). Measurements taken over time on degradation or accumulated stress contain information about the reliability of units. There are several works in the literature, on the modeling of degradation leading to failure. Gorjian et.al (2010: [39]) and Nikulin et.al (2010: [68]) are examples of the papers in this direction. In many of these models it is common to choose a fixed threshold $s(t) = s$ for the degradation $X(t)$ for all $t \geq 0$. The reliability at time t for these models is given by $R(t) = P[X(t) < s]$. Even with data from a relatively small number of units, one can hope to achieve better specification of reliability by harnessing this information (see, [20]). Similar studies in this area, were done in [93], [24], [22], [98], [96] and [97].

Murray (1993: [69]) initially performed the degradation data analysis by setting models to sample path for the individual units and obtained pseudo failure times. Further, these failure times were analyzed using common life data analysis methods. Meeker et.al. (2009: [64]) took the random effects model to describe the unit-to-unit variability and that showed how a degradation model along with a failure definition, induces a failure time distribution. Marta. A. Freitas et al. (2009: [33]) made it clear by presenting three classical methods: namely, analytical, numerical and approximate methods to estimate the failure time distribution of degradation models $D(t) = \beta t$ and $D(t) = (1/\beta)t$. They used various parametric distributions such as Weibull, Lognormal and normal distributions for the random parameter. Illustrations were also presented as a case study on train wheel degradation data. Julio C. Fereira et.al (2012: [32]) discussed the case study on train wheel degradation data by taking $D(t) = \alpha_0 + e^{\eta} t$ as degradation model with η specifying the random effect parameter. Time to failure distribution of wheels is obtained based on the position of wheels. Later, Freitas et al. (2010: [34]) conducted a similar study by presenting five methods of degradation data analysis. Parametric degradation models by considering linear degradation paths and simple non-linear degradation paths, are studied by authors in [8], [58] and [40]. The non-linearity nature (increasing/decreasing) of measurements can be seen in real life examples like train wheel degradation data and drug potency degradation data.

Degradation was modeled as a function of time t by Lu and Meeker (1993: [20]). The function is given by $X(t) = \mu(t, \bar{\theta}, \bar{\phi}),\ t \geq 0$, where $\bar{\phi}$ is a vector of fixed effect parameters, $\bar{\theta}$ is a vector of random effect parameters and the degradation is measured with additive error at specified times. In their model, the event of a critical crack length exceeding a constant level of 1.6 inches is defined as a failure (that is, $s(t) = 1.6$ inches). They had considered a data set consisting of fatigue crack length measurements at equi-spaced time points for many metallic specimens under test.

Thus the degradation path is known for some systems, and in such cases, useful information on the reliability of a product can be obtained from these degradation measurements. As a contribution to estimation using degradation data, in this chapter, a degradation model having exponential degradation path with positive degradation rate, which follows a Weibull distribution with known shape parameter and unknown scale parameter is considered. The corresponding unknown parameters are estimated. Baye's estimate of scale parameter of Weibull distribution is also obtained, and thereby Bayesian reliability of first kind and second kind for the system are computed.

The rest of the chapter is organized as follows. Section 6.2 describes the general degradation path model, the exponential degradation model and illustrates the reliability estimation methodology. In Section 6.3, the Bayesian estimation of reliability under informative and noninformative priors are presented. Section 6.4 contains the bootstrap method for finding standard error of Bayes estimator of α, with respect to both informative and noninformative priors. Section 6.5 contains the Gibbs sampling procedure for estimating reliability. Section 6.6 provides the numerical illustration and Section 6.7 gives the conclusions.

6.2 General degradation path model

As an example of degradation modeling, Lu and Meeker (1993: [20]) introduced the General Path Model to move reliability analysis methods from time-of-failure analysis to process-of-failure analysis. The observed degradation, that is the degradation of $i-th$ unit at time t_j, is given by

$$y_{ij} = \eta_{ij} + \epsilon_{ij},\ \ i = 1, 2, ..., k;\ \ j = 1, 2, ..., n,$$

where $\eta_{ij} = \eta(t_j,\ \Phi,\ \Theta_i)$, n is the number of inspections, t_j is the time of the $j-th$ measurement, ϵ_{ij} is the measurement error with constant variance, η_{ij} is the actual path of the $i-th$ unit at time t_j, Φ is the vector of fixed-effect parameters, common for all units, Θ_i is the vector of the $i-th$ unit random-effect parameters, representing individual unit characteristics. Θ_i and ϵ_{ij} are assumed to be independent of each other.

6.2.1 Exponential degradation model

Freitas et.al (2010: [35]), have considered a linear degradation path model for train wheels degradation, to obtain their lifetime distribution. Bae and Kvam (2008: [91]) considered the exponentially decreasing degradation path. Gebraeel et.al (2005: [37]) developed a Bayesian updating method that uses real-time condition monitoring data to update the stochastic parameters of an exponential degradation model. In this chapter, a strictly increasing degradation path is considered and is given by

$$y(t) = ae^{\theta t}, \ a > 0, \ \theta > 0, \tag{6.2.1}$$

where a is the initial degradation, θ is the random parameter representing the rate of degradation and t is the time. Many real life situations can be modelled by using strictly increasing degradation path. For example, the train wheel degradation data considered in [35], is an example of such kind.

6.2.2 Description of reliability function

Assume that the rate of degradation parameter, θ in the model given by equation 6.2.1 described above, follows Weibull distribution with unknown scale parameter α and known shape parameter β. Then, the probability density function of θ is given by

$$f(\theta) = \alpha\beta\theta^{\beta-1}e^{-\alpha\theta^{\beta}}, \ \alpha > 0, \ \beta > 0 \text{ and } \theta > 0. \tag{6.2.2}$$

The cumulative distribution function is then given by

$$F(\theta) = 1 - e^{-\alpha\theta^{\beta}}.$$

Let s be the threshold value, then the reliability at time t is described as $R(t) = Pr(y(t) < s)$. Observe that, this definition of reliability make sure that the amount of degradation at time t, does not exceed the threshold s. When the degradation crosses s, then the system is categorized as a failed one (see [91]). That is, $R(t) = Pr\left(ae^{\theta t} < s\right), \ s > a$, by using equation 6.2.1. Then $R(t)$ can be computed as $R(t) = Pr\left(\theta < \frac{log(s/a)}{t}\right), \ t > 0, \ log(s/a) > 1$.
Since θ follows Weibull distribution, the reliability function is given by

$$R(t) = 1 - e^{-\alpha\left(\frac{log(s/a)}{t}\right)^{\beta}}; \ t, \ \alpha, \ \beta > 0 \text{ and } \frac{s}{a} > 1. \tag{6.2.3}$$

It is easy to verify that, when $t \to 0$, $R(0) = 1$ and when $t \to \infty$, $R(\infty)$ goes to 0.

6.3 Bayesian estimation procedure

In this section, a brief explanation of Bayesian estimation procedure, is presented. Suppose θ follows Weibull distribution with parameters $(\alpha,\ \beta)$. Possible values of the parameter α are inferred by assuming that the uncertainty about α can be expressed by a probability density function $\pi(\alpha)$, which is called the prior density. This prior density or distribution of a parameter is the probability distribution that represents uncertainty about the parameter before the examination of current data. Assuming that the sample is generated by a conditional probability density $f(\theta \mid \alpha)$, a sample of observations is collected to learn more about α.

In the light of the given data on θ, the uncertainty about α is to be updated. That is, to find the conditional distribution of α when θ is given. Using Bayes theorem, the posterior distribution of the parameter is obtained by

$$P(\alpha \mid \theta) = \frac{L(x \mid \alpha, \theta)\pi(\alpha)}{\int L(x \mid \alpha, \theta)\pi(\alpha)\ d\alpha},$$

where $L(x \mid \alpha, \theta)$ is the likelihood function of the given random sample, $x = (x_1, x_2, \cdots, x_n)$. A prior distribution is considered noninformative, if compared to the likelihood function, it is relatively flat. Thus, $\pi(\alpha)$ is noninformative, if it has minimal impact on the posterior distribution of α. An informative prior is that which is not dominated by the likelihood function and which has an impact on the posterior distribution. If the likelihood function is dominated by the prior distribution, then the prior is certainly an informative one. A proper use of prior distributions illustrate the power of the Bayesian method: information gathered from the previous study, past experience, or expert opinion can be combined with current information in a natural way (see, [41]).

In practical situations, like in case of measuring the strength of a concrete, one can measure the strength of the concrete made at any time t, given that cracks on the concrete are developing as per the law $y = ae^{\theta t}$. For example, rate of degradation (θ) of the concrete can be assumed as per the exponential law. That is, $\theta \sim E(\lambda)$, where λ is the degradation per unit time. Further, suppose that we have measured data on degradation at specified time points, then the knowledge on degradation can be modelled using appropriate prior distribution, say $\pi(\alpha, \beta)$ for λ, and hence one can obtain Bayes estimate of λ, and thereby corresponding system reliability. In the following section, Bayesian system reliability by considering increasing degradation path model given by equation 6.2.1, is obtained.

For Bayesian estimation of $R(t)$, noninformative and informative prior probability distributions for α are considered. For these two priors, the first kind of Bayesian reliability estimator and the second kind of Bayesian reliability estimator are obtained (see, for example, [6]).

6.3.1 Bayesian estimation of reliability using noninformative prior

Assume that the rate of degradation θ follows Weibull distribution with parameters α and β, then the probability density function of θ is given by

$$f(\theta) = \alpha\beta\theta^{\beta-1}e^{-\alpha\theta^{\beta}}, \; \alpha > 0, \; \beta > 0 \text{ and } \theta > 0, \tag{6.3.1}$$

where α is the unknown scale parameter and β is the known shape parameter. Let the noninformative prior for the parameter α follow Quasi density function given by $\pi(\alpha) = \frac{1}{\alpha^k}$, $k > 0$. Next, consider the following theorem which will give posterior distribution of α given θ, when noninformative Quasi prior is considered for α.

Theorem 6.3.1. *The posterior distribution of the parameter α given θ follows Gamma distribution with parameters $n - k + 1$ and $\sum_{i=1}^{n} \theta_i^{\beta}$, where θ follows Weibull $(\alpha, \; \beta)$.*

Proof: The posterior distribution of α is given by

$$P(\alpha \mid \theta) = \frac{L(x \mid \alpha, \theta)\pi(\alpha)}{\int L(x \mid \alpha, \theta)\pi(\alpha)\; d\alpha}$$

$$= \frac{\alpha^n\beta^n \prod_{i=1}^{n} \theta_i^{\beta-1} e^{-\alpha \sum_{i=1}^{n} \theta_i^{\beta}} \frac{1}{\alpha^k}}{\int_0^{\infty} \alpha^n\beta^n \prod_{i=1}^{n} \theta_i^{\beta-1} e^{-\alpha \sum_{i=1}^{n} \theta_i^{\beta}} \frac{1}{\alpha^k}\; d\alpha}$$

$$= \frac{\alpha^{n-k}}{\Gamma(n-k+1)} \left(\sum_{i=1}^{n} \theta_i^{\beta} \right)^{n-k+1} e^{-\alpha \sum\limits_{i=1}^{n} \theta_i^{\beta}}.$$

This implies $P(\alpha \mid \theta)$ follows Gamma distribution with parameters $n-k+1$ and $\sum\limits_{i=1}^{n} \theta_i^{\beta}$.

The Bayesian estimator $\hat{\alpha}$ of α using squared error loss function is obtained by minimizing the mean square error $E\left((\hat{\alpha}-\alpha)^2\right)$. That is, consider the following:

$$E\left((\hat{\alpha}-\alpha)^2\right) = \int (\hat{\alpha}-\alpha)^2 \pi(\alpha)\, d\alpha.$$

Then $\frac{d}{d\hat{\alpha}} E\left((\hat{\alpha}-\alpha)^2\right) = 0$ gives $\hat{\alpha} = E(\alpha \mid x)$.

That is, the posterior mean is the required Bayesian estimator of α. The expectation of α from posterior distribution $P(\alpha \mid \theta)$ is then the required Bayesian estimator. That is,

$$\hat{\alpha} = \frac{n-k+1}{\sum\limits_{i=1}^{n} \theta_i^{\beta}}.$$

The Bayesian estimator of reliability of first kind $\hat{R}_1(t)$ is obtained by replacing α by $\hat{\alpha}$ in expression given by equation 6.2.3. Thus

$$\hat{R}_1(t) = 1 - e^{-\hat{\alpha}\left(\frac{log(s/a)}{t}\right)^{\beta}}.$$

Hence,

$$\hat{R}_1(t) = 1 - e^{-\left(\frac{n-k+1}{\sum\limits_{i=1}^{n} \theta_i^{\beta}}\right)\left(\frac{log(s/a)}{t}\right)^{\beta}}.$$

The Bayesian estimator of reliability of Second kind $\hat{R}_2(t)$ is obtained by

$$\begin{aligned} \hat{R}_2(t) &= E\left(R(t) \mid \alpha\right) \\ &= \int_0^{\infty} \left(1 - e^{-\alpha\left(\frac{log(s/a)}{t}\right)^{\beta}}\right) \left(\frac{\alpha^{n-k}}{\Gamma(n-k+1)} \left(\sum_{i=1}^{n} \theta_i^{\beta} \right)^{n-k+1} e^{-\alpha \sum\limits_{i=1}^{n} \theta_i^{\beta}} \right) d\alpha. \end{aligned}$$

(See, [6] for more details).

Hence,

$$\hat{R}_2(t) = 1 - \frac{\left(\sum_{i=1}^{n} \theta_i^{\beta}\right)^{n-k+1}}{\left(\sum_{i=1}^{n} \theta_i^{\beta} + \left(\frac{log(s/a)}{t}\right)^{\beta}\right)^{n-k+1}}.$$

6.3.2 Bayesian estimation of reliability using informative prior

Assume that the informative prior for the parameter α follow Gamma distribution with parameters c and d. Then,

$$\pi(\alpha) = \frac{d^c}{\Gamma(c)} \alpha^{c-1} e^{-d\alpha}, \ c > 0 \text{ and } d > 0.$$

Next, consider the following theorem which will give posterior distribution of α given θ, when informative Gamma prior is considered for α.

Theorem 6.3.2. *The posterior distribution of the parameter α given θ follows Gamma distribution with parameters $n + c$ and $d + \sum_{i=1}^{n} \theta_i^{\beta}$, where θ has the density given by equation 6.3.1.*

Proof: The posterior distribution of α is given by

$$P(\alpha \mid \theta) = \frac{L(x \mid \alpha, \theta)\pi(\alpha)}{\int L(x \mid \alpha, \theta)\pi(\alpha) \ d\alpha}.$$

After incorporating the expression of $\pi(\alpha)$ and, evaluating the above expression, note that

$$P(\alpha \mid \theta) = \frac{\alpha^n \beta^n \prod_{i=1}^{n} \theta_i^{\beta-1} e^{-\alpha \sum_{i=1}^{n} \theta_i^{\beta}} \frac{d^c}{\Gamma(c)} \alpha^{c-1} e^{-d\alpha}}{\int_0^{\infty} \alpha^n \beta^n \prod_{i=1}^{n} \theta_i^{\beta-1} e^{-\alpha \sum_{i=1}^{n} \theta_i^{\beta}} \frac{d^c}{\Gamma(c)} \alpha^{c-1} e^{-d\alpha} \ d\alpha}.$$

Simplification of above expression gives

$$P(\alpha \mid \theta) = \frac{\left(d + \sum_{i=1}^{n} \theta_i^{\beta}\right)^{n+c}}{\Gamma(n+c)} \alpha^{n+c-1} e^{-\alpha\left(d + \sum_{i=1}^{n} \theta_i^{\beta}\right)}. \tag{6.3.2}$$

This implies $P(\alpha \mid \theta)$ follows Gamma distribution with parameters $n+c$ and $d+\sum_{i=1}^{n} \theta_i^{\beta}$.

Using squared error loss function, the posterior mean is the Bayesian estimator of the parameter α. It is easy to verify that

$$\hat{\alpha} = \frac{n+c}{d+\sum_{i=1}^{n}\theta_i^{\beta}}.$$

The Bayesian estimator of reliability of first kind, $\hat{R}_1(t)$ is obtained by substituting $\hat{\alpha}$ in equation 6.2.3. That is,

$$\begin{aligned}\hat{R}_1(t) &= 1-e^{-\hat{\alpha}\left(\frac{log(s/a)}{t}\right)^{\beta}} \\ &= 1-e^{-\left(\frac{n+c}{d+\sum_{i=1}^{n}\theta_i^{\beta}}\right)\left(\frac{log(s/a)}{t}\right)^{\beta}}.\end{aligned}$$

The Bayesian estimator of reliability of Second kind $\hat{R}_2(t)$ is obtained by taking expectation of $R(t)$, using posterior distribution of α in equation 6.3.2. Thus,

$$\begin{aligned}\hat{R}_2(t) &= E\left(R(t)\mid\alpha\right).\text{ That is,} \\ \hat{R}_2(t) &= \int_0^{\infty}\left(1-e^{-\alpha\left(\frac{log(s/a)}{t}\right)^{\beta}}\right)\left(\frac{\left(d+\sum_{i=1}^{n}\theta_i^{\beta}\right)^{n+c}}{\Gamma(n+c)}\alpha^{n+c-1}e^{-\alpha\left(d+\sum_{i=1}^{n}\theta_i^{\beta}\right)}\right)d\alpha \\ &= 1-\frac{\left(d+\sum_{i=1}^{n}\theta_i^{\beta}\right)^{n+c}}{\left(d+\sum_{i=1}^{n}\theta_i^{\beta}+\left(\frac{log(s/a)}{t}\right)^{\beta}\right)^{n+c}}.\end{aligned}$$

6.4 Bootstrap standard error of $\hat{\alpha}$

The bootstrap method to estimate standard error of $\hat{\alpha}$ can be used if the distribution of $\hat{\alpha}$ is unknown or complicated (see, [30]). In this chapter, the degradation rate θ follows Weibull distribution with parameters α and β. Here, n denote the number of samples taken from Weibull distribution, $c,\ d$ are the hyper parameters of α, where α follows Gamma distribution, and B is the bootstrap sample size. The bootstrap method to find the standard error of $\hat{\alpha}$ is given below.

Step 1: Read the values of $n,\ \alpha,\ \beta.\ c,\ d$ and B.

Step 2: Set $i = 1$

Step 3: Generate n random samples $\theta_1.\ \theta_2,\ \cdots,\ \theta_n$ from Weibull distribution with parameters α and β.

Step 4: Estimate $\hat{\alpha}(i)$ from random sample obtained in Step 3.

Step 5: Using the estimate $\hat{\alpha}(i)$, generate a bootstrap sample with size n from the Weibull distribution with parameters $\hat{\alpha}(i)$ and β.

Step 6: $i = i + 1$

Step 7: $\alpha = \hat{\alpha}(i)$

Step 8: if $i <= B$ go to Step 3, else go to Step 9.

Step 9: The average bootstrap estimate $\bar{\hat{\alpha}} = \frac{\sum_{i=1}^{B} \hat{\alpha}}{B}$.

Step 10: The bootstrap standard error $S_{\hat{\alpha}} = \sqrt{\frac{\sum_{i=1}^{B} (\hat{\alpha} - \bar{\hat{\alpha}})^2}{B-1}}$.

6.5 Gibbs sampling procedure for estimating reliability

Gibbs sampling is a Markov Chain Monte Carlo (MCMC) algorithm for obtaining a sequence of observations which are approximated from a specified probability distribution, when direct sampling is difficult. This Gibbs sampling is commonly used as a means of statistical inference, especially Bayesian inference. It is a randomized algorithm, and is an alternative algorithms for statistical inference such as the expectation-maximization algorithm (EM). In this section, Gibbs sampling procedure to obtain Bayesian system reliability estimation is given in following steps:

Step 1: Assume initial value for the parameter θ. Let it be θ^0.

Step 2: Let $j = 1$.

Step 3: Generate random value for the parameter θ^j from the obtained distribution of the parameter.

Step 4: Compute the system reliability R^j for the value of the parameter obtained in Step 3.

Step 5: Set $j = j + 1$.

Step 6: Repeat the Steps 2 to 5 for M number of times.

Step 7: Obtain the approximate posterior mean of system reliability

$$\hat{R} = \frac{1}{M}\sum_{j=1}^{M} R^j.$$

6.6 Illustrations through numerical examples

In this section, for the selected inputs, the Bayesian reliability estimate using the degradation model $y = ae^{\theta t}$, for the case of informative prior and noninformative prior, are illustrated. The Bayesian reliability estimate of first and second kind obtained from Gibbs sampling procedure is plotted in figures from 6.1 to 6.40.

6.6.1 Calculation of bootstrap standard error of $\hat{\alpha}$

Let $n = 20$ be the number of observed Weibull samples with shape parameter $\beta = 0.7$ and B denotes the bootstrap sample size. Let the initial value of the scale parameter of Weibull distribution $\alpha = 0.9$, and $\bar{\hat{\alpha}}$ be the average bootstrap estimate of α. The bootstrap standard error of estimated α is presented in the following Table 6.1, Table 6.2 and Table 6.3 with respect to various combination of hyper parameters, for the case of informative prior as well as noninformative prior. It is observed that standard error of $\hat{\alpha}$ decreases as the bootstrap sample increases. Then $\hat{\alpha}$ behaves as a minimum variance estimator for moderately large bootstrap sample. Thus one can obtain an Bayesian estimator of first kind using this minimum variance estimator $\hat{\alpha}$, which is practically desirable. Let SE denote the standard error in the following tables.

Table 6.1: Standard error of $\hat{\alpha}$ for Informative Prior

B	$\hat{\alpha}$ for $c = 1,\ d = 5$	SE for $c = 1,\ d = 5$	$\hat{\alpha}$ for $c = 3,\ d = 4$	SE for $c = 3,\ d = 4$
100	0.8427	0.1843	0.9940	0.2711
200	0.8331	0.1739	0.9659	0.2108
300	0.8303	0.1679	0.9800	0.2068
400	0.8287	0.1671	0.9921	0.2036
500	0.8354	0.1647	0.9938	0.2026
600	0.8368	0.1625	0.9888	0.2003
700	0.8233	0.1565	0.9824	0.1990
800	0.8134	0.1524	0.9847	0.1963
900	0.8111	0.1459	0.9886	0.1952
1000	0.8176	0.1458	0.9799	0.1903

Table 6.2: Standard error of $\hat{\alpha}$ for Noninformative Prior

B	$\hat{\alpha}$ for $k = 1$	SE for $k = 1$	$\hat{\alpha}$ for $k = 2$	SE for $k = 2$
100	1.0535	0.3329	1.0277	0.3214
200	1.0905	0,3187	1.0058	0.2799
300	1.0740	0.2881	0.9874	0.2757
400	1.0822	0.2871	0.9934	0.2644
500	1.0653	0.2816	0.9793	0.2579
600	1.0618	0.2800	0.9896	0.2556
700	1.0555	0.2792	0.9963	0.2524
800	1.0687	0.2755	1.0169	0.2511
900	1.0863	0.2703	0.9922	0.2489
1000	1.0667	0.2639	0.9790	0.2442

Table 6.3: Standard error of $\hat{\alpha}$ for Noninformative Prior

B	$\hat{\alpha}$ for $k = 8$	SE for $k = 8$	$\hat{\alpha}$ for $k = 15$	SE for $k = 15$
100	0.6249	0.1813	0.2011	0.0606
200	0.6206	0.1748	0.2028	0.0572
300	0.5943	0.1683	0.1928	0.0547
400	0.5851	0.1569	0.1995	0.0540
500	0.5890	0.1550	0.1933	0.0521
600	0.5915	0.1546	0.1952	0.0509
700	0.5803	0.1503	0.1925	0.0499
800	0.5848	0.1489	0.1912	0.0498
900	0.5791	0.1462	0.1949	0.0489
1000	0.5787	0.1446	0.1941	0.0475

6.6.2 Bayesian reliability estimate based on informative prior

For the informative Gamma prior with parameters c and d, estimates of system reliability for increasing sample size $n = \{20, 50, 100, 200\}$ and $M = 500$ are plotted for four different sets of input parameters in following figures (figures 6.1 to 6.8). These figures are respectively correspond to the input sets $I_1,\ I_2,\ I_3$ and I_4 given in Table 6.4. In each of these, it can be observed that the estimated system reliability approaches the actual reliability as the sample size increases.

Table 6.4: Inputs for informative prior

	I_1	I_2	I_3	I_4
a	0.9	0.8	0.25	1
β	0.5	0.45	0.3	0.05
s	2	1.5	2.5	0.9
c	20	18	21	10
d	5	8	3	1.5

6.6.2.1 Bayesian reliability estimate of first kind $R_1(t)$

In this section, the Bayesian reliability estimate of the first kind using informative prior is plotted for the sample sizes $n = 20,\ 50,\ 100,\ 200$ corresponding to the inputs given in Table 6.4 (See figures 6.1 to 6.4).

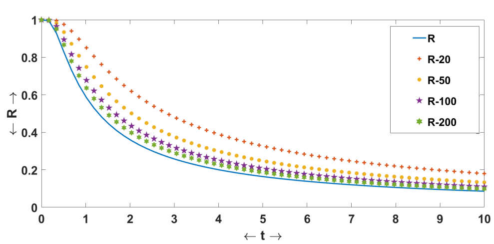

Figure 6.1: $\hat{R}_1(t)$ corresponding to input I_1

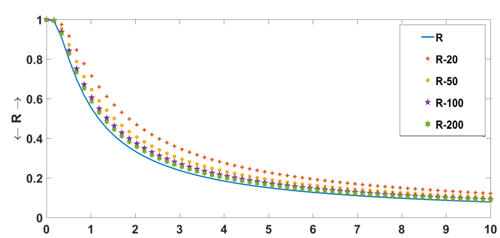

Figure 6.2: $\hat{R}_1(t)$ corresponding to input I_2

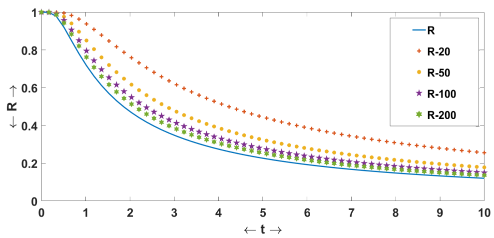

Figure 6.3: $\hat{R}_1(t)$ corresponding to input I_3

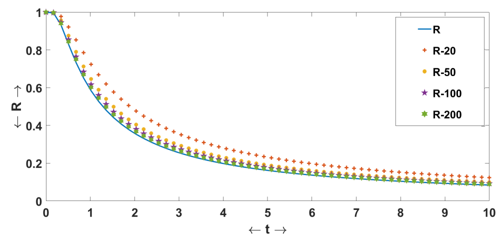

Figure 6.4: $\hat{R}_1(t)$ corresponding to input I_4

6.6.2.2 Bayesian reliability estimate of Second kind $R_2(t)$

In this section, the Bayesian reliability estimate of the second kind using informative prior is plotted for the sample sizes $n = 20,\ 50,\ 100,\ 200$ corresponding to the inputs given in Table 6.4 (See figures 6.5 to 6.8).

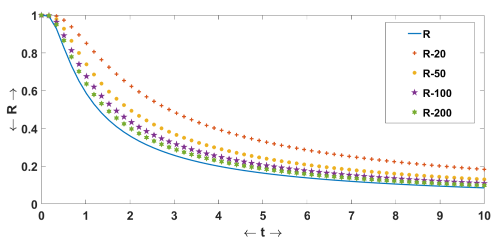

Figure 6.5: $\hat{R}_2(t)$ corresponding to input I_1

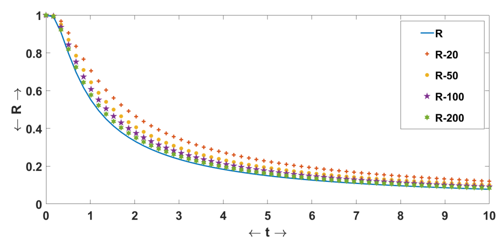

Figure 6.6: $\hat{R}_2(t)$ corresponding to input I_2

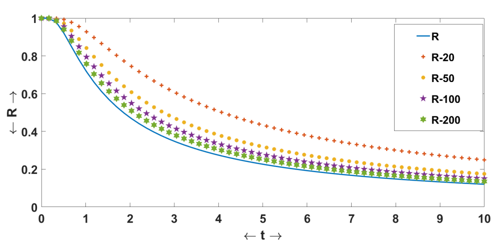

Figure 6.7: $\hat{R}_2(t)$ corresponding to input I_3

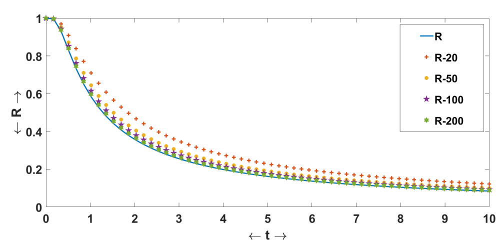

Figure 6.8: $\hat{R}_2(t)$ corresponding to input I_4

6.6.3 Bayesian reliability estimate based on noninformative prior

For the noninformative Quasi prior with $k = 1, 2, 8$ and 15, consider the inputs given in Table 6.5 below:

Table 6.5: Inputs for noninformative prior

	I_1	I_2	I_3	I_4
a	0.9	0.8	0.25	1
β	0.5	0.45	0.3	0.05
s	2	1.5	2.5	0.9

6.6.3.1 Bayesian reliability estimate of first kind $R_1(t)$

In this section, the Bayesian reliability estimate of the first kind using noninformative prior is plotted for the sample sizes $n = 20,\ 50,\ 100,\ 200$ corresponding to the inputs given in Table 6.5 (See figures 6.9 to 6.24).

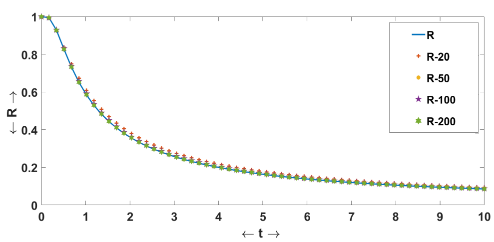

Figure 6.9: $\hat{R}_1(t)$ for the input I_1, k=1

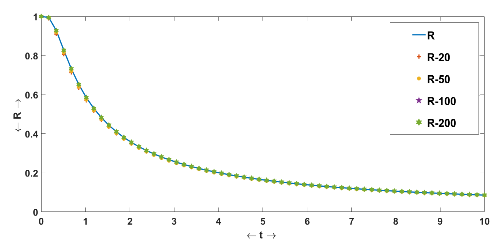

Figure 6.10: $\hat{R}_1(t)$ for the input I_1, k=2

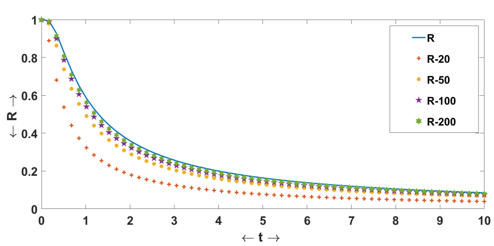

Figure 6.11: $\hat{R}_1(t)$ for the input I_1, k=8

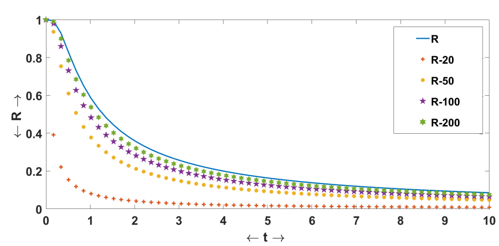

Figure 6.12: $\hat{R}_1(t)$ for the input I_1, k=15

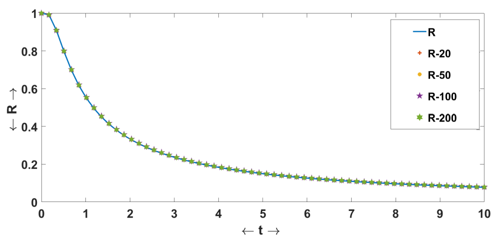

Figure 6.13: $\hat{R}_1(t)$ for the input I_2, k=1

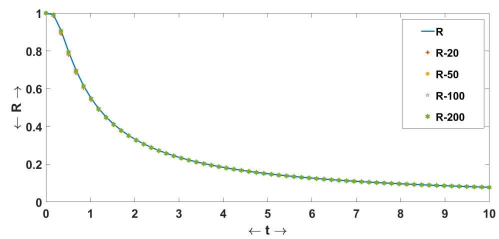

Figure 6.14: $\hat{R}_1(t)$ for the input I_2, k=2

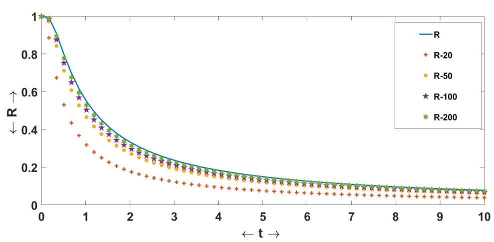

Figure 6.15: $\hat{R}_1(t)$ for the input I_2, k=8

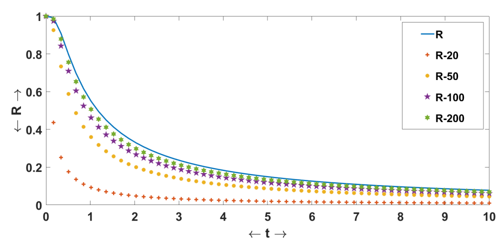

Figure 6.16: $\hat{R}_1(t)$ for the input I_2, k=15

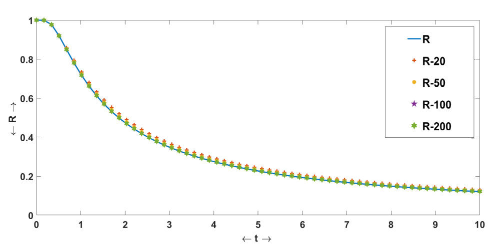

Figure 6.17: $\hat{R}_1(t)$ for the input I_3, k=1

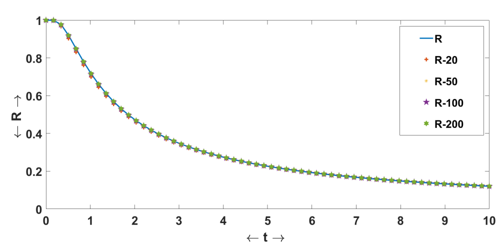

Figure 6.18: $\hat{R}_1(t)$ for the input I_3, k=2

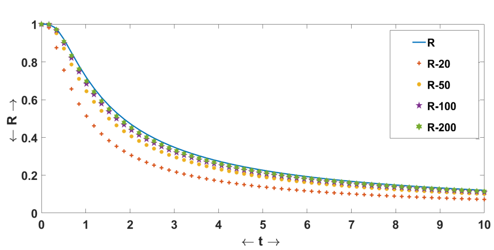

Figure 6.19: $\hat{R}_1(t)$ for the input I_3, k=8

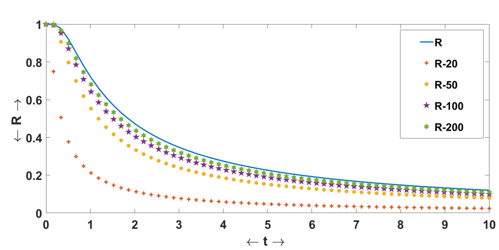

Figure 6.20: $\hat{R}_1(t)$ for the input I_3, k=15

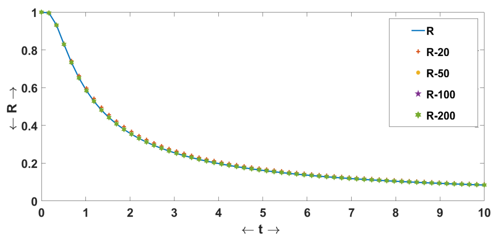

Figure 6.21: $\hat{R}_1(t)$ for the input I_4, k=1

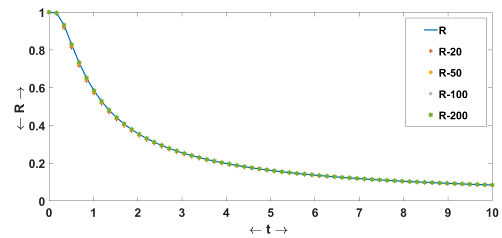

Figure 6.22: $\hat{R}_1(t)$ for the input I_4, k=2

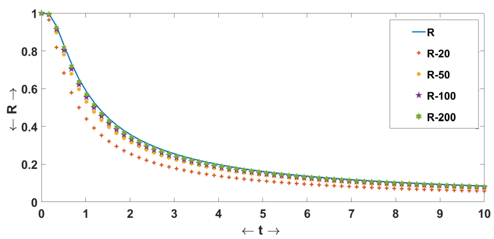

Figure 6.23: $\hat{R}_1(t)$ for the input I_4, k=8

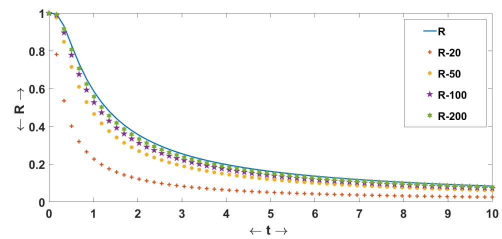

Figure 6.24: $\hat{R}_1(t)$ for the input I_4, k=15

6.6.3.2 Bayesian reliability estimate of Second kind $R_2(t)$

In this section, the Bayesian reliability estimate of the second kind using noninformative prior is plotted for the sample sizes $n = 20,\ 50,\ 100,\ 200$ corresponding to the inputs given in Table 6.5 (See figures 6.25 to 6.40).

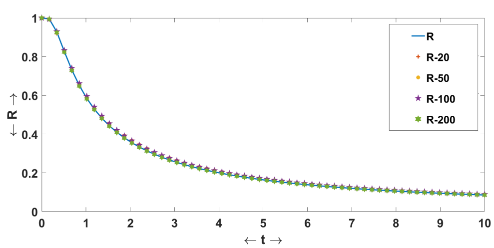

Figure 6.25: $\hat{R}_2(t)$ for the input I_1, $k = 1$

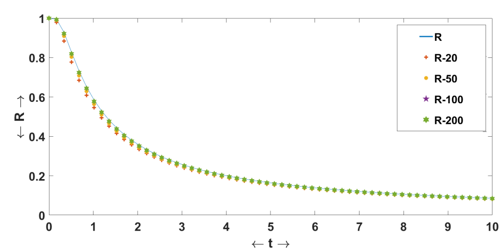

Figure 6.26: $\hat{R}_2(t)$ for the input I_1, $k = 2$

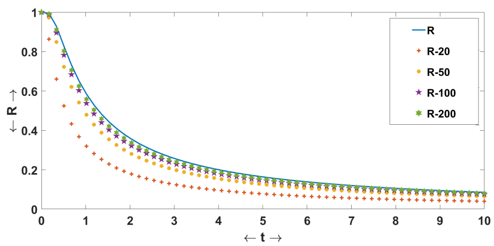

Figure 6.27: $\hat{R}_2(t)$ for the input I_1, $k = 8$

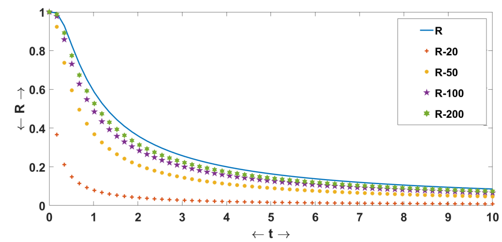

Figure 6.28: $\hat{R}_2(t)$ for the input I_1, $k = 15$

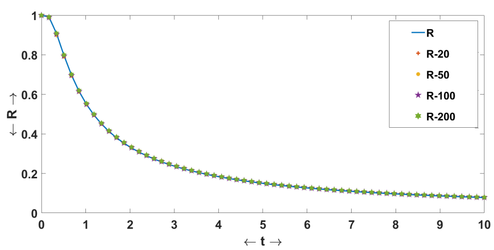

Figure 6.29: $\hat{R}_2(t)$ for the input I_2, $k = 1$

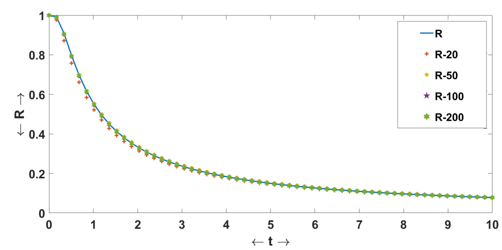

Figure 6.30: $\hat{R}_2(t)$ for the input I_2, $k = 2$

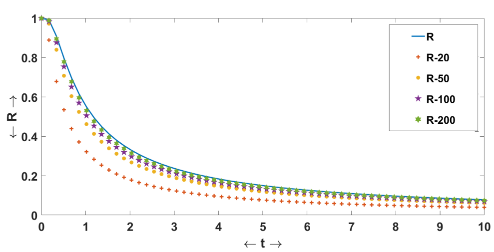

Figure 6.31: $\hat{R}_2(t)$ for the input I_2, $k = 8$

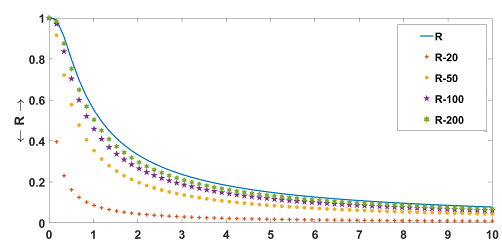

Figure 6.32: $\hat{R}_2(t)$ for the input I_2, $k = 15$

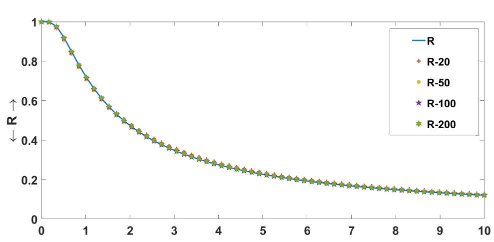

Figure 6.33: $\hat{R}_2(t)$ for the input I_3, $k = 1$

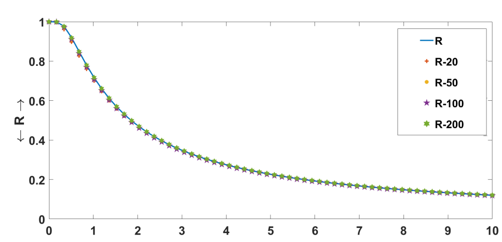

Figure 6.34: $\hat{R}_2(t)$ for the input I_3, and $k = 2$

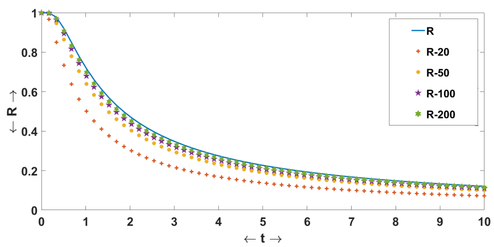

Figure 6.35: $\hat{R}_2(t)$ for the input I_3, $k = 8$

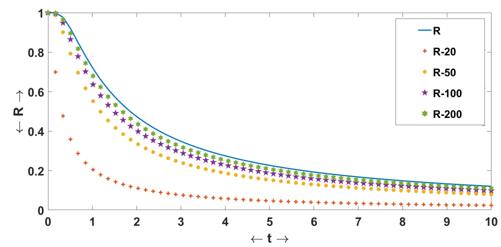

Figure 6.36: $\hat{R}_2(t)$ for the input I_3, $k = 15$

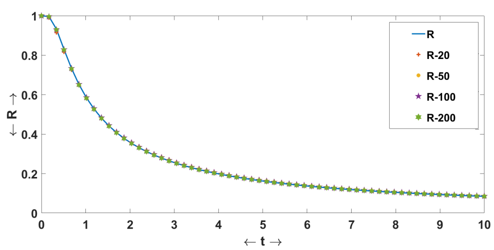

Figure 6.37: $\hat{R}_2(t)$ for the input I_4, $k = 1$

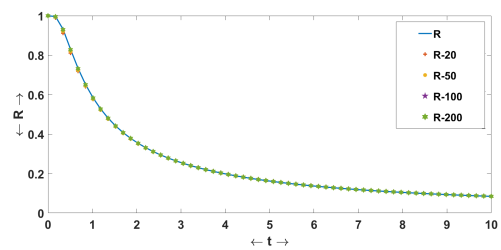

Figure 6.38: $\hat{R}_2(t)$ for the input I_4, $k = 2$

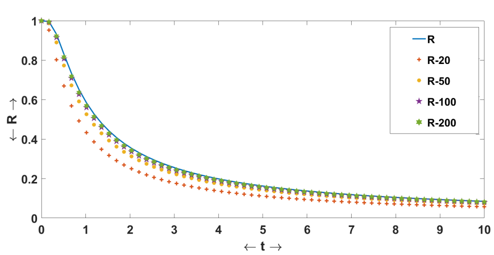

Figure 6.39: $\hat{R}_2(t)$ for the input I_4, $k = 8$

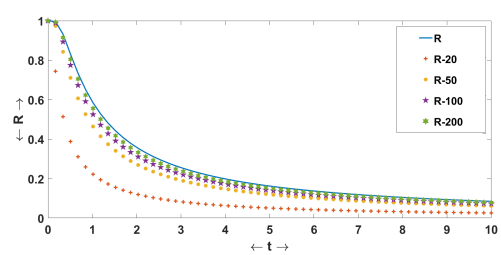

Figure 6.40: $\hat{R}_2(t)$ for the input I_4, $k = 15$

6.7 Conclusions

The Bayesian estimator for the scale parameter of θ is evaluated by assuming noninformative and informative prior distributions and, thereby the corresponding estimators of reliability are obtained in Sections 6.3.1 and 6.3.2. The following observations are made:

a) The posterior distribution for the scale parameter α under noninformative Quasi prior follows Gamma distribution with parameters $n-c+1$ and $\sum_{i=1}^{n}\theta_i^{\beta}$.

b) The posterior distribution for the scale parameter α under informative Gamma prior follows Gamma distribution with parameters $n+c$ and $d+\sum_{i=1}^{n}\theta_i^{\beta}$.

In both informative and noninformative prior cases, it can be observed from figures, 6.1 to 6.40, that the estimated system reliability approaches the actual reliability when the sample size increases.

In the case of Quasi prior, note that if $k=1$, we will get the maximum likelihood estimator of system reliability. In all four sets of inputs in the Table 6.5, when $k=1$, the estimated reliability is almost the same as the original reliability.

In both informative and noninformative cases, the standard error of $\hat{\alpha}$ is decreasing when the bootstrap sample size increases. Thus one can have a Bayesian reliability estimator by using minimum variance Bayes estimator of α.

Chapter 7

Conclusions and Scope for Future Research

The major contributions to this thesis are (a) In existing reliability test plans, based on two predefined constants $0 < R_0 < R_1 < 1$, the system is subjected for its reliability test, and it will be accepted if R, the reliability for unit time, exceeds R_1, and the same will be rejected whenever $R \leq R_0$. As an initial contribution, an alternate criterion for testing the reliability of a system, namely, an acceptable reliability interval (ARI) and unacceptable reliability interval (URI) are defined in testing the reliability of a parallel system. Thus a small relaxation of an amount ε to ARL is applied while accepting the system. On the other hand, the URL is increased by an amount ε, for checking the unacceptability of the system. The same concept can be extended to a series system as well. This criterion has some advantages in reducing the huge burden of rejection cost. (b) In testing the reliability of a series system based on data obtained from Type-II censoring, a normal cost of testing is considered by several authors. But the cost of testing under Type-II censoring need not be constant; in fact, the cost associated is a random quantity, which is very difficult to deal with. An attempt is made in the thesis, to design reliability test plan for a series system, by minimizing the associated maximum-total-expected-testing-cost under Type-II censoring. (c) Based on data available on the failure rate of components in a system, suitable prior distribution can be fitted, and the same can be incorporated with underlying density function to derive Bayes estimate for failure rates, and thereby corresponding system reliability. As a next contribution, Bayesian reliability test plans for parallel and series systems are designed. These plans have advantages in savings in testing costs of the system as compared to that with classical test plans. (*d*) As obtaining data under normal working conditions of a system is more difficult, one may end up with a small amount of failure data, or there is a need to wait for a long period to get the required amount of failure data. In such situations, an accelerated test is the most convenient option. Thus, an attempt is made to derive the optimal reliability sampling plan based on data obtained from partially accelerated life test using two types of life-stress relations, namely, linear and Arrhenius relationships. (*e*) Usual practice of obtaining failure data is performing life tests on units in a given lot. These tests may be sometimes destructive and may cause huge financial loses. One alternative to avoid such loses is making use of readily available degradation data, as such data encloses information on failures of the system. Thus an attempt is made to make use of such degradation data, by considering exponential degradation growth model with positive degradation rate (θ, which is a random variable) to obtain reliability estimate of the desired system, as the last contribution in this thesis. The following conclusions were made in each of the chapters.

In Chapter 2, an optimal reliability test plans for parallel systems with failure rate as the exponential function of covariates are designed. The data are obtained through Type-II censoring scheme. Unbiased estimator and maximum likelihood estimator of the failure rate of the individual component in the system, are used to estimate system reliability, and corresponding optimal reliability test plans are designed. In the case of reliability test plan constructed using unbiased estimator of failure rate, the resulting reliability test plan has about 60% reduction in total testing costs. In the case of reliability test plan constructed using maximum likelihood estimator of system reliability, definitions for satisfactory and unsatisfactory levels of system reliability are introduced as an alternative to acceptable reliability level (ARL) and unacceptable reliability level (URL) respectively, and they are called acceptable reliability interval (ARI) and unacceptable reliability interval (URI). Introduction of these relaxed definitions have some advantages in reducing testing costs or huge rejection cost incurred in the usual testing scenario (of using strict ARL and URL criterion). Several numerical examples are illustrated and compared with the existing results. It is observed that there is a significant reduction in testing costs of about 70%.

Chapter 3 presents the reliability test plan for a series system. First, reliability test plan is constructed using the system reliability estimate obtained from unbiased estimator of failure rate of individual components in the system. Through this work, it is shown that for a series system, the optimum design depends on the cost of individual components and that all components need not be tested equally. Unlike most of the plans available in literature, in the proposed plan, the acceptance constant d^* and the optimum sample size for each component depend upon the testing costs of individual components. Also, it is observed that no test plan in the literature uses prior information in the form of upper bound which is a function of covariates. However, through the proposed plan, it is observed that use of prior information and incorporation of covariates have advantage of about 79% savings in testing costs as illustrated by examples. Secondly, maximum likelihood estimator of failure rates of components in the system are used to obtain MLE of system reliability. The maximum-total-expected-testing-cost expression is obtained. Since it is difficult to minimize random testing cost involved in Type-II censoring, an optimization problem is formulated to minimize maximum-total-expected-testing-cost and optimal parameters are obtained subjected to the requirements of Type-I and Type-II error constraints. The results available in the literature (for example, [82] and [73]), propose a test plan for series system, where, (a) one has to test all components equally irrespective of cost of the component, and the number of components in the system, (b) the failure rate is a constant, (c) the testing cost ia a fixed constant, (d) it is not necessary to solve the optimization problem for the general n-component series system, instead it is sufficient to solve the problem for one component system, (e) the acceptance constant d^* and the optimum sample size for each component are not depending upon testing cost. In this work, it is observed for a series system, that (i) optimum design depends upon cost of the individual component and number of components in the system, (ii) the cost is not considered as a constant under Type-II censoring, instead it is considered as a function of time, and thereby maximum-total-expected-testing-cost involved in testing the entire series system is minimized to obtain optimal plan parameters, (iii) the problem is solved for a general n component series system, (iv) the acceptance constant d_M^* and the optimum sample size for each component are depending upon testing cost. A simulation study conducted shows that in the proposed model, the derived sampling plans actually meet the specified risks α and β. Also, from the sensitivity analysis, it is clear that the model is sensitive in its parameters and corresponding output is stable. A qualitative analysis is also done. Incorporation of covariate information in modeling failure rates of components as linear combination of covariates and, considering the testing cost as a function of time have significant advantage in reducing the total number of components to be tested for failure. Moreover, this type of testing the reliability of a system by obtaining data under Type-II censoring, has an advantage of obtaining realistic results, since the system is tested under normal working conditions. It is observed that the percentage of components to be tested for failure is reduced by about 96%. Also, there is a significant reduction in testing costs of about 77% as compared to that in [82], and 96% as compared to that in [73].

Chapter 4 discusses the design of optimal reliability test plan for a series and parallel system with failure rates of components as random variables having Quasi-density. The data are obtained through Type-I censoring scheme, and the reliability estimator is obtained by estimating a Bayesian estimator of component failure rates. Some numerical examples are also computed to illustrate the Bayesian approach of estimating system reliability and thereby to test the system reliability. The proposed Bayesian plan has about 70% savings in total testing costs.

Optimal design of reliability acceptance sampling plans based on data obtained from partially accelerated life test using linear and Arrhenius life-stress relationships are presented in Chapter 5. The Type-II censoring scheme is used to obtain the required data. The Maximum likelihood estimates of the unknown parameter of

Weibull distribution and acceleration factor are obtained for the linear model. Similarly, MLEs of model parameters are obtained in case of the Arrhenius model as well. Several examples are presented to illustrate our optimal acceptance test plans. It is observed that the test cost involved in constructing an acceptance sampling plan using PALT is random. Hence an expression for expected testing cost is given and the same is illustrated through several examples. However, the actual cost involved in testing may be less than that reported in this work. It is observed that when the values of the producer's and consumer's risks decrease, the testing cost increases. Also, the total expected testing cost is less in case of the plan obtained using the Arrhenius model, and it is about 93% less as compared to that in the case of the linear model. It is observed that Arrhenius life-stress model is more cost-effective than the linear life-stress model.

Finally, in Chapter 6, degradation growth model is considered to estimate system reliability using data collected from degradation measurements. Bayes estimate of scale parameter α of Weibull distribution of degradation parameter θ (rate of degradation) is obtained. Bayesian reliability of first kind and second kind for the system are computed under informative and noninformative priors. The bootstrap method is used for finding standard error of Bayes estimator of α, with respect to both informative and noninformative priors. Gibbs sampling procedure is used for estimating reliability. The following observations are made: (i) The posterior distribution for the scale parameter α under noninformative Quasi prior follows Gamma distribution with parameters $n - c + 1$ and $\sum_{i=1}^{n} \theta_i^{\beta}$. (ii) The posterior distribution for the scale parameter α under informative Gamma prior follows Gamma distribution with parameters $n + c$ and $d + \sum_{i=1}^{n} \theta_i^{\beta}$. In both informative and non-informative prior cases, it is observed that the estimated system reliability approaches the actual reliability when sample size increases. In the case of Quasi-prior, if $k = 1$, we will get the maximum likelihood estimator of system reliability. It is also observed through a numerical computation that, the estimated reliability is almost the same as the original reliability when $k = 1$. In both informative and non-informative cases, the standard error of $\hat{\alpha}$ is decreasing when the bootstrap sample size increases. Thus one can have a Bayesian reliability estimator by using minimum variance Bayes estimator of α.

The future works shall include open problems to construct the reliability test plan for a general series/parallel system under Type-I/Type-II censoring, with the assumption that the components in the systems are dependent. Also, the test plans considered in this thesis for a series system with Type-II censoring (using random testing cost) can be extended to a parallel system by considering testing cost as a random quantity. The acceptance sampling plan using data from step-stress partially accelerated life test for Weibull distribution can be obtained by considering the notion of minimizing random testing costs subjected to the requirements of Type-I and Type-II error constraints.

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LIST OF PUBLICATIONS

Journal papers/Book chapter/Conference papers

1 M Kumar and P N Bajeel. *Design of Component Reliability Test Plans for a Series System Having Time Dependent Testing Cost with the Presence of Covariates*. Computational Statistics. Springer-Verlag Germany. Vol. 33 (3), pp. 1267-1292, (2018) - **SCI & Scopus** indexed.

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