CHAPTER 10
A Mathematical Model for Human Hearing Based on Energy Consideration

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1. INTRODUCTION

The purpose of this paper is to study how the ear perceives the sound waves and responds to sound of different frequencies. Basically, we target on the three objectives: detailed study of the sound wave propagation through auditory system, develop a mathematical model for describing the perceived sound wave by human hearing and solve the model and detailed analysis on mathematical theory of least square approximation for human hearing. Mainly, this study implicated the sound wave propagation in hearing and applies the method of least square approximation to a model for human hearing and is motivated by energy consideration.

1.2 History of hearing research

The evolution of hearing began quite a while back alongside the mammalian development. Inception in development of warm blooded creatures originally found in the Triassic period around 230 years ago, during this period, fostered a tympanic middle ear in all land vertebrates. Advancement of Hearing is followed by the advancement in the ear structures that began in the dryolestes (late Jurassic mammals) and proceeded to its current state. In particular, dryolestes were the blend for familial uncoiled cochlea and neomorphic bony cochlear from which the ear structures of marsupials and placentals are evolved. Currently, Therians (placental and marsupial warm blooded creatures), cochlea curling evolves and high frequency hearing with completely looped cochlea seen first in cretaceous. These new changes are connected with the earliest expansion of metatherians and eutherians in the Cretaceous, prompting incremental hearing in Cenozoic and residing marsupials and placentals.

The history of hearing research can be traced back to ancient times, when philosophers and physicians like Aristotle and Hippocrates speculated about the nature of hearing and the mechanisms behind it. However, it wasn't until the 16th and 17th centuries that scientists began to conduct systematic experiments to study hearing. Modern hearing exploration began in the sixteenth century; Andreas Vesalius named the two ossicles; malleus and incus while his scholar Philippus Ingrassia coined the third ossicle, stapes in 1546. The cochlea was found by Bartholomeus Eustachius in 1552 and the name 'cochlea' was given by Gabriel Falloppio in 1561. In the seventeenth century, Italian researcher Giovanni Battista Porta found the cochlea, the twisted and coiled structure in the inner ear that assumes to be the vital part in hearing. Thomas Willis estimated in 1672 that each "tones" may stimulate various fibres of the nervous acoustics. In a joint effort with the physicist Edme Mariotte, Joseph Guichard Duverney proposed theory of the tonotopical association of the cochlea in 1683, which stated the encoding of acoustic data by mechanical spectral analysis. In the 18th century, French physician Rene Laennec invented the stethoscope, which allowed doctors to listen to sounds within the body, including sounds associated with hearing. The academic doctrine of Aristotle's ear implantation was refuted by Domenico Cotugno in 1760.

In the 19th century, several significant discoveries were made in the field of hearing research. In 1800, Alessandro Volta discovered the electric battery, which led to the development of electrical stimulation as a tool for studying the nervous system. This laid the foundation for the use of electricity in hearing research. In the mid- 1800s, German physicist Hermann von Helmholtz developed the theory of hearing that proposed that different frequencies of sound are detected by different parts of the inner ear, thus ear detects sound through the movement of tiny hair cells in the cochlea. Duverney's hypothesis was, along with Georg Simon Ohm's law on the approach of sound waves with Fourier analysis, which served as the premise of Hermann Helmholtz' renowned hypothesis of hearing in 1863. Owing to the absence of comprehensive information about the nerve functions, still couldn't find any reasonable thoughts on the acoustical data conveyed to the brain by the acoustic nerve. The French researcher Edouard Leon Scott de Martinville discovered a gadget called the phonautograph, which can record sound. Organ of sorti was found by Alphonso Corti and was named after him by Albert von K¨olliker, and the functions of vestibular organs were tested and showed through Flouren's tests. Besides, the ground works of auditory psycho-physics were done by Alfred M. Mayer and others.

In the 20th century, advances in technology allowed researchers to study hearing in even greater detail. In the early 1900s, researchers began to investigate the relationship between sound and the nervous system. One of the major breakthroughs during this time was the discovery of the cochlear amplifier by Georg von Beksey, which showed that the cochlea has an active mechanism that amplifies sounds. American scientist Harvey Fletcher developed the first electronic hearing aid in 1920, which used vacuum tube amplifiers to amplify sound. In the 1940s, researchers at Bell Labs developed the first cochlear implant, a device that stimulates the auditory nerve directly and can provide hearing to people with severe hearing loss. In the mid-20th century, the development of new technologies such as the audiometer, which measures hearing sensitivity, and the cochlear implant, which can restore hearing to those with severe hearing loss, revolutionized the field of hearing research. Scientists also began to focus on the neural processes involved in hearing, leading to significant advances in our understanding of how the brain processes sound. Recently, research has concentrated on the genetic and molecular basis of hearing, similarly on the improvement of new treatments for hearing impairment, including stem cell therapy and gene therapy. Today, hearing research continues to be a rapidly evolving field with significant implications for improving the lives of those with hearing impairments. Furthermore, the research on various species had provided with enough information on the mechanism of hearing. The queries on the function of auditory system in human can be cleared by choosing species which allows the study of hearing mechanism easy when compared to human. Obviously, this methodology of comparison has aided the auditory neuroscience well.

A few instances of fundamental examination in non-mammalian species that have steered the field are:

1. Research on turtles, frogs, and birds found that their hair cells endure more effectively in vitro than mammalian hair cells which laid foundation to the study of biophysics of hair cells. (reviewed in Hudspeth 2014).
2. Comprehension in sound localization enhanced after the study conducted in owls.(Grothe et al., 2018)
3. The importance of auditory feedback in speech production is evident from the study that the degradation of birdsong is slowed down when their auditory feedback is intruded. (See Mooney, 2018).
4. The early exposure to noise spur up hearing loss with respect to increase in age, was revealed from the study on mice. (Kujawa and Lieberman, 2019).
5. Restoration of hair cells in birds unfolded the possibility that hair cells can be regenerated in mammals (reviews in Fettiplace, 2020).
6. To study the genetics of hearing impairment, mice and zebra fish are the best. (e.g., Vona et al., 2)

1.3 Research focus

In this paper, we develop a mathematical model for describing the distinguished sound waves by human hearing based on energy consideration. The sound waves are entering to the outer ear and move along the ear canal to the middle ear. The ear canal carries the waves to eardrum, causing it to vibrate. These vibrations continue into the inner ear through inter connected apparatuses that lead to the cochlea. The cochlea contains a liquid that is altered by the various sound waves. The alterations are picked up by tiny hair cells. Furthermore, these hair cells are automatically connected to the ion channels of distinct neurons in the brain, and these neurons are stimulated by the Fourier transformation of the sound waves. That is, our ear is an effective transformer. Scientifically, in the inner ear, the basilar membrane consist of distinct frequencies, the excessive frequencies compose a massive vibration at the end; closest to the middle ear and moderate frequencies has a extensive vibration along the faraway. Thus, the human auditory system can analyze the frequencies and it will be same as to Fourier transformation.

In the ear, the movements of the tiny hairs cause to form the acoustic energy in the fluid of cochlea. This energy is accurately found by the calculating the error between two respective functions; periodic sound wave $p(t)$ with period T and a complex sinusoidal sound wave $q(t)$. However, there is some sound wave $q(t)$ and periodic sound wave $p(t)$ produces the same sensation of sound.

To solve the mathematical model, we use the least square approximation. The least squares model for human hearing refers to a mathematical method for fitting a curve to data that aims to minimize the sum of the squares of the error between the observed data and the corresponding estimated curve. In the context of human hearing, this method is used to describe a model for relationship between the physical stimulus of sound and the perceived loudness by a listener. The goal of the least squares model is to find the best-fitting curve that accurately predicts the subjective loudness of different sound stimuli, and this information can be used in various applications such as sound engineering and psycho-acoustics.

1.4 Literature Review

Helmholtz (1954) accomplished an incredible work on the anatomy of the ear. Helmholtz proposed a resonance theory of hearing in 1863. The cochlear partition was considered as a series of tuned resonators by this theory. Here, high tones situated at the base while low tones at the peak. While responding to a single tone, the cochlear partition would vibrate just in the confined locale which can resonate that specific tone. It is similar to the functioning of a piano, with damper raised, whose strings will vibrate specifically to single acoustic tones. This resonance theory must be modified with the arrival of von Bekesy’s perceptions on the movement of the cochlear partition.

The experiments done and perceptions made on human corpses, animals, and models which started in 1928 by G. von Bekesy's laid the foundation to our understanding on the human ear mechanism. And it helped him bag a Nobel Prize in 1961. Von Bekesy's formulated a method for estimating the displacement of the cochlear partition with respect to a sinusoidal tone in 1947. The cochleae of animals and human corpses were examined by Von Bekesy's under a microscope with the help of stroboscopic illumination. Fig.3.1 is an example of von Bekesy's test results from a human cochlea. He then noticed the motion of waves on the basilar membrane; waves which move up the cochlea with rising amplitude and lower wavelength. He found that these waves at some positions are reaching a maximum level and afterward fall away quickly. The area of maximum displacement differs in frequency as said in resonance theory. The waves with high frequencies are only moving a small distance whereas waves with low frequencies move up to the helicotrema.

Von Bekesy's observations triggered Peterson and Bogert to design a long- wave model in 1950. They believed that waves spread in the cochlea were long compared to the cross-sectional diameter and made a resemblance to waves on an electrical transmission line. They decided the firmness of the basilar film from They used von Bekesy's test-hair measurements to estimate the firmness of the basilar membrane and utilized the fluid mass inside the cochlear channel as the partition mass. They expected components of the basilar membrane to have no coupling in the longitudinal direction. They found a numerical solution for the movement of basilar membrane by ignoring fluid viscosity and partition damping.

Ranke (1950) assumed that waves in the cochlea were small near to the characteristic place whereas long in the basal area. He formulated a short- wave model in which he expected the wavelength to be small all over and the fluid is inviscid and impossible to compress. He then compared his outcomes with those of von Bekesy.

Siebert (1974) designed a two-dimensional model expecting the fluid to be inviscid and impossible to compress. His design was in the form of an integral equation. Later, he connected his model to the long wave model of Zwislocki (1965) and the short-wave model of Ranke (1950). Prior to tracking down answers for his model, Siebert made a short-wave approximation.

Alkahby et al proposed two models for the basilar membrane, particularly mathematical model, in 1999. One model depicted basilar membrane in a shape of ring whiles the other as a rectangular region. The results thus obtained stipulated the significance of curvature of basilar membrane in hearing mechanism.

The round window membrane vibrates with an opposite phase to acoustic vibrations entering the cochlea through the stapes at the oval window, another opening of the cochlea to the middle ear.

Through this chapter, we precisely explained the basic history and researching methodologies in human hearing mechanism and briefly gave an explanation of the purpose this paper with its aims and objectives. And also, we take a short analyzing on literature review related to the field of technological and physiological aspects of the sound mechanism and illustrated the contributions on study of sound mechanisms.

2. SOUND CONDUCTION MECHANISM

We precisely illustrate about conduction mechanism of auditory system with the amplifying main roles of outer ear and middle ear. The Conduction is the arrival of sound waves from an external source to the outer ear which passes through the middle ear before reaching the inner ear.

Outer ear: The outer ear plays a major role in sound conduction by transmitting sound waves from outside the ear to the tympanic membrane. The pinna helps in collecting and passing the sound waves to the ear canal. The angle in which pinna collects the sound, it helps the individual to locate the source. But it is only possible for a sound wave having higher frequency, because of the head size and the wavelength of wave, and in middle and lower frequencies, sound shadow projected by head itself and time taken for a wave to reach ear helps in sound localization

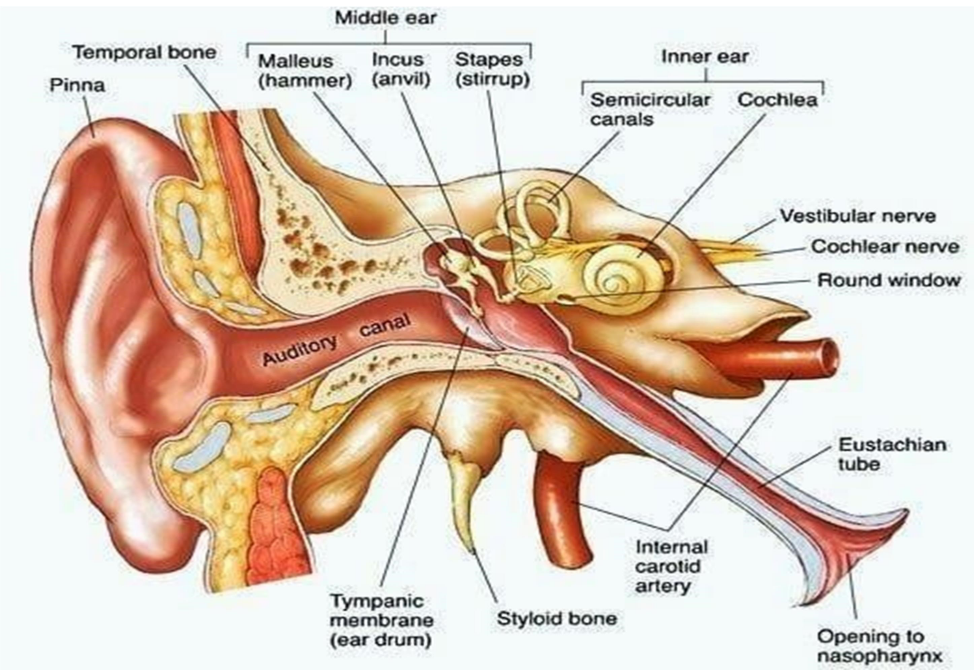

Fig 2.1: A cross section of the human auditory system. (Source: Encyclopedia Britannica)

respectively. The 4 cm long auditory canal has hairs and ear wax in it which acts as a disinfectant. The external layer of the tympanic membrane itself is formed of skin same as that of the auditory canal.

The skin in body of organisms has the ability of restoration. The skin of auditory canal falls off into the wax of external ear. This is why we shouldn't use cotton buds to clean the ear, because it affects hearing. The curve in ear canal prevents entry of external particles to tympanic membrane. The tympanic membrane is the integral part in sound transduction which has a shape of loudspeaker cone. The tympanic membrane is called eardrum by many, but actually, middle ear is eardrum while tympanic membrane is drum skin.

Middle ear: The air-occupied space that is linked with Eustachian tube is called middle ear. The middle ear contains three small bones namely - malleus, incus and stapes which carries sound from tympanum to inner ear. Tympanic membrane serves as the outer boundary of the middle ear while cochlea acts as inner wall.

The middle ear enfolds jugular bulb, vein that controls cerebral blood flow. Eustachian tube opens to the middle ear and mastoid air cells at front and posterior ends respectively. Muscle contraction helps in balancing the pressure in the middle ear and nose well. The malleus has a shape of club where its top lies in middle ear cavity above tympanum. The cone shaped incus has its base at the head of the malleus. The bend at its tip connects to the stapes. The stapes is third bone which links the middle and inner ear. Its foot plate enfolds the opening to the inner ear, named oval window.

Generally, in this chapter we analyzed the mechanism of sound conduction. We obtained detailed information about the major roles of outer ear and middle ear in the conduction process.

3. SOUND TRANSDUCTION MECHANISM

We illustrate about transduction mechanism of auditory system with the amplifying main role of inner ear. The transduction is the converting of sound waves into electrical signals and they transmit to the brain.

Inner ear: In the inner ear, snail-shell shaped cochlea play major role in sound transduction, i.e., it is core of the interpretation of sound waves. The hearing organ is called membranous labyrinth enveloped by fluid-perilymph. The cochlea consists of 0.2 milliliter volume; where located millions of hair cells and nerve fibres. The hair cells are also named stereocilia. The surface of sensory cells in the stereocilia consists of microscopic, hair like protrusions, and they are the sensory receptors of auditory and vestibular systems of the ear. The hair cells transduce vibrations into nerve signals and then abundant nerve fibres that act as a channel for signal transmission.

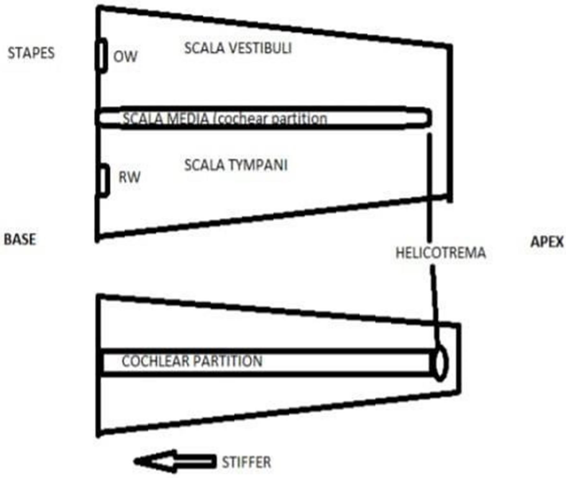

Fig 3.1: The cochlea is a bony tube, filled with perilymph in which floats the endolymph filled membranous labyrinth (Hallowell and Silverman, 19, 1970)

The uncoiled portion of cochlea is a bony tube with round and oval windows, so there is an association between membranous labyrinth and vestibular labyrinth. The Vibrations in the stapes foot plate causes to vibrate the fluid-perilymph within the cochlea. Consequently, it causes to opening of oval window and round window. The membranous labyrinth consists of three components; the outermost two components are scala vestibuli and it is linked to the oval window and the other component scala tympani are linked to round window. The helicotrema is the minute slit that link the perilymph filled portion at their tip, In the case of frequencies below the audible range it role as pressure equalizing mechanism. Moreover, they link to the brain by a passage called perilymphatic aqueduct. The membranous labyrinth filled by a fluid endolymph and is also called cochlear duct. They also separated one side by reissner's membrane from scala vestibuli and other side by the basilar membrane from scala tympani. The basilar membrane consists of large number of nerve fibres, and it is trembled by the vibrations of the hair cells with the sound waves then the transduction of vibrations with the audible range to a nerve signal. The basilar membrane reacts as vibrantly to greater frequencies at the lowest end of oval window and to gradually smaller frequencies as forward movement to highest end. The basilar membrane reacts vibrantly to the lower frequencies at the highest end. The impulses are formed as wave at the oval window and are channeled to the basilar membrane, and those nerve fibres are vibrating at gradually provide the transmission of information to brain. As a result we sensed the sound. Then the cochlea is a notable well regulated frequency analyzer.

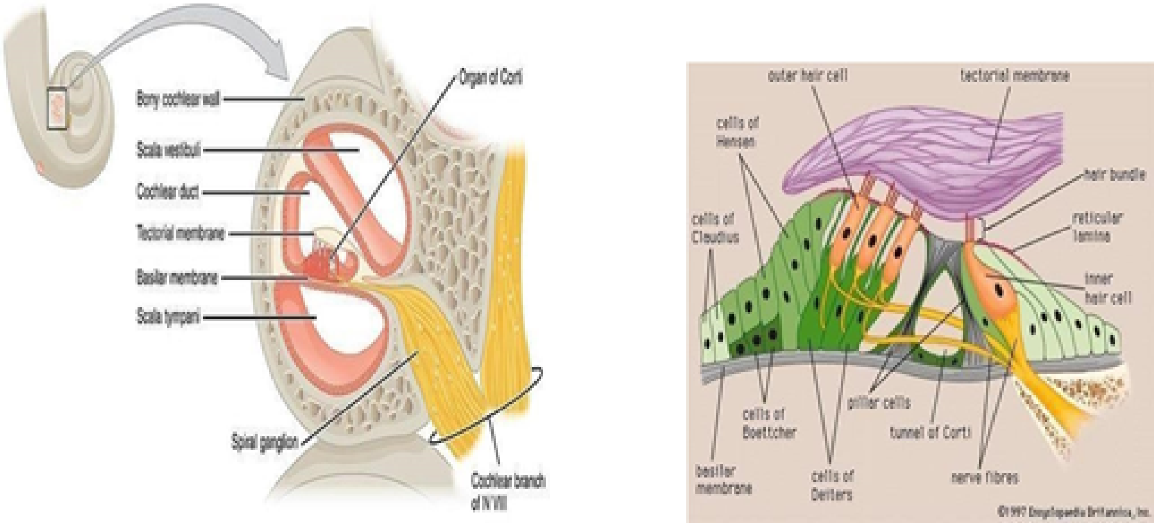

Figure 3.2: A cross section of cochlea showing details of membranous labyrinth with organ of corti. (Source: Encyclopedia Britannica)

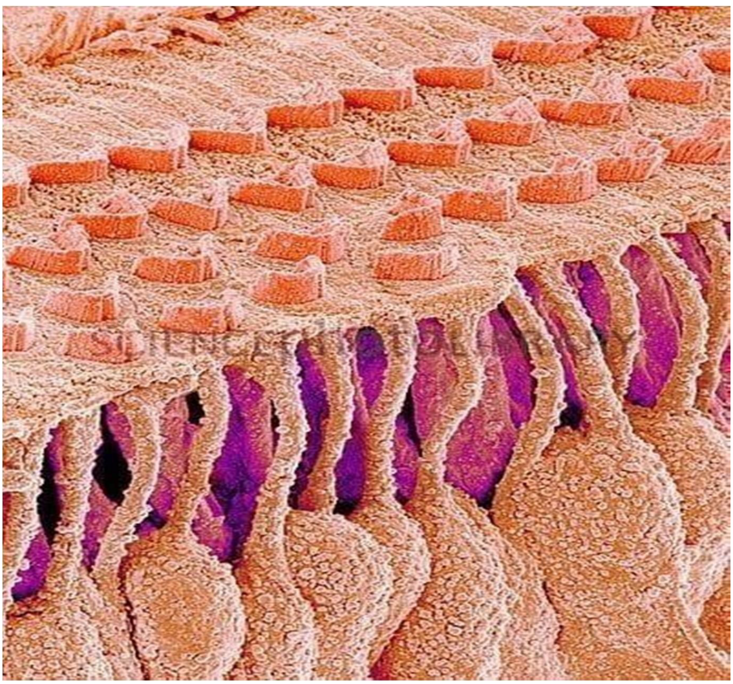

Figure 3.3: A surface view of the hair cells in the cochlea (Source: sciencephoto library)

Anatomically, the human auditory system has the mechanisms of sound conduction and sound transduction. In the sound conduction mechanism, there are two basic parts; the pinna and ear canal within the outer ear and tympanic membrane within the middle ear. The Eustachian tube attached to nose from the air space of middle ear, then also linked to the ossicular chain, consists of three tiny bones; the malleus incus and stapes. In the cochlea, transduce the vibrations and channeled to the perilymph along the ossicular chain into nervous signals hence pass to the brain then it is sensed as sound.

4. HEARING MECHANISM

Here deals with the discussion about; how the auditory system performs in the mechanism of hearing within the audible range, and also illustrate the contribution of outer, middle and inner ears.

Outer and Middle ears: Let's talk about the conduction mechanism the human being can hearing the sounds within the range of 20 Hz to 20000 Hz. The ear cans extreme sensitive in the range of 128 Hz to 4000 Hz. when the increasing of age the auditory and sensitivity of humans are decreasing. The human head serve as a natural hindrance between those two sided ears. The ear can feel extreme stimulus, when the sound source closest to it. Those sound waves are varying with change of time and it promote to localizing of sound for hearing mechanism. The wrinkled shape of pinna helps to seize the more intense frequencies sound and then passes to ear canal. The pinna can protect the ear from the arrival of some high frequencies sound waves; as a result it helps to recognize the source of sound. The ear along this sounds serve as resonating tube so it causes to more intense vibrations of sound. The ear has an important role in the balancing of human body by equalizing of pressure. In the ear, Eustachian tube acts as pressure equalizer. The sound signals are amplified by the outer and middle ears. The collected sound signals pass to the ear drum and then to ossicular chains within the middle ear. The ear drum act as a hydraulic amplifier. The ossicular chain also helps to boosting of sound signals.

Inner ear: The approaching vibrations are transducing to nerve signals by the inner ear. Nerve fibres of the basilar membrane start to vibrate and it causes to mobility of molecules in the fluid perilymph. It produces an acousting signals and transport to brain. Nerve fibres can vibrate under the range of 200 Hz. The frequency range above kHz cannot audible clearly. The basilar membrane consists of distinct levels of frequencies on different ends. The motions of ossicular chain causes to collect the vibrations by the oval window and it transport to basilar membrane and all frequencies attain their respective position of vibration; and there diminishes their movements. The hair cells of cochlea have separation by tunnel of corti into one row of inner hair cells and three rows of outer hair cells. We can sense the nerve signals as sound after the initiation of inner hair cells.

The inner hair cells are sensitive towards higher sound frequencies and outer hair cells are sensitive towards the lower frequencies. The amplified sounds are transmitting to inner hair cell from outer hair cells. Then the inner hair cells are initiated to the nerve signals. The human ear is extreme sensitive to sound in the range of 3000 Hz to 4000 Hz. The high stimulus production within this range of frequency, it causes to damage of outer hair cells. As a result, the inner hair cells cannot sense the lower frequencies and it also reason to the hearing loss at slight frequencies. But still can hear the sounds of high frequency. This phenomenon is called as loudness recruitment.

4.1 Summary

The sound waves coming to ear are collected and passed into tympanic membrane by pinna in outer ear. It creates vibrations of air particles which is then transmitted to tympanum, eventually tympanum starts to vibrate. The vibrations in tympanic membrane prompt the three small bones in ear to vibrate. The third small bone which is near to the inner ear, stapes causes movement in oval window. The oval window poses a smaller surface compared to tympanic membrane, so the vibrations in the oval window are intensified around twenty times. The movement in oval window induces the perilymph in the cochlea to vibrate. Since the perilymph is an incompressible fluid, it creates motion in round window. Atmospheric air and air in middle air comes in contact through Eustachian tube and pharynx. There is a membrane in between upper canal and cochlea. This membrane moves according to vibrations in perilymph. This push endolymph to and fro which in turn led to the vibration of basilar membrane, but the tectorial membrane remains static. Subsequently, the hair cells between tectorial and basilar membrane and enables auditory nerve in transmission of these impulses to cerebrum, the largest part of brain. The basilar membrane near round window vibrates more with the arrival of higher frequency sound waves whereas the basilar membrane near to the tip of cochlea when a lower frequency wave comes. The brain is thus able to recognize the frequency of sound wave from where it is arrived from.

Range of Hearing: Normally, the human ear can respond to sound with the frequency of 40 to 16 000 Hz. Animals like dogs and bats can hear very high frequency sounds which our which our ears cannot detect. As we grow older, we lose the ability to hear higher pitched sounds. The dB (decibel) is the measured unit of loudness. As sound that can barely be heard, is given a decibel rating of zero. The whispering sound is the range of 30dB and normal communication around the 60 dB. The high frequency noises above 120 dB can harm the ears.

This is because they exert a great pressure on the tympanic membrane.

5. LEAST SQUARE MODELLING IN HUMAN HEARING MECHANISM

We apply the least squares approximation method in the modelling of human hearing, and it is motivated by energy consideration. Linear algebra techniques are used to approximate a function over an interval by a linear combination of sine and cosine terms.

5.1 Introduction

Let's start with a short analysis about the idea of sound and human hearing. By showing in the figure 5.1, it is the schematic diagram of the ear and its three basic parts: the outer ear, middle ear and inner ear. The sound waves are entering to the outer ear where they are diverted to the eardrum, make happen it to vibrate. Three small bones in the middle ear precisely connect to the ear drum with the snail- shaped cochlea inside the inner ear. These bones transmit the vibrations of the ear drum to a fluid inside the cochlea. The cochlea contains millions of tiny hairs that vibrate with the fluid. As a result the area near to entrance of the cochlea is stimulated by high frequencies, and the area near to the top of the cochlea is stimulated by low frequencies. The fluctuation of these hairs actuates nerve cells that convey signals along different neural pathways to the brain, where the signals are decoded as sound.

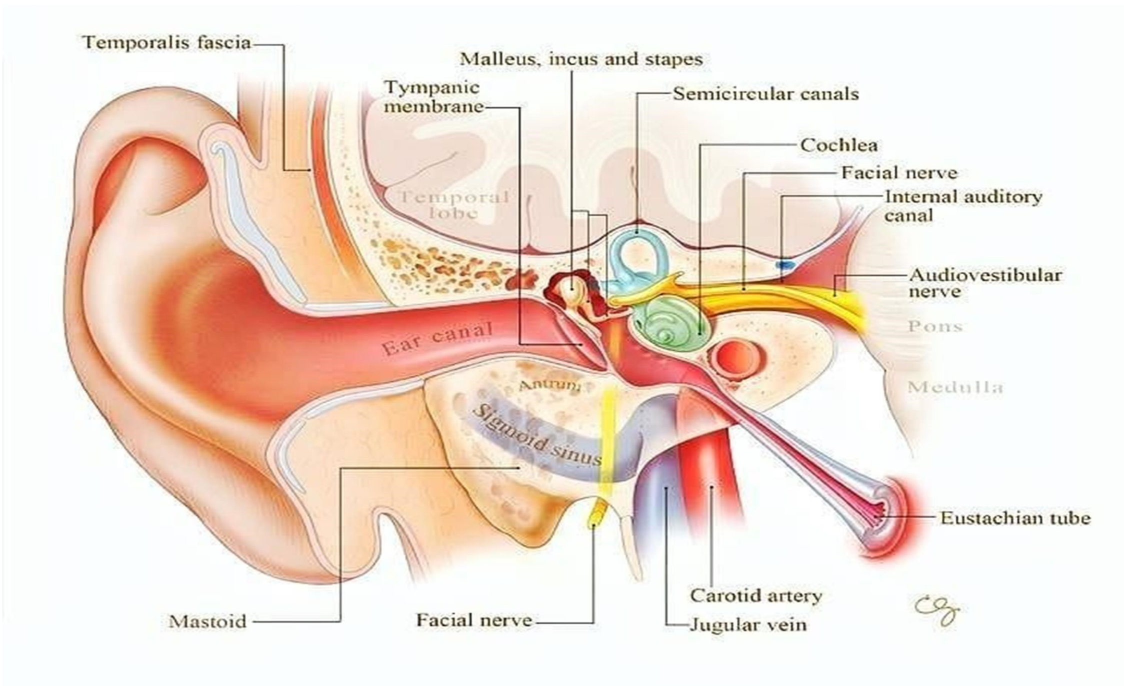

Fig 5.1

5.2 Mathematical Model Formulation

The sound waves are the consequence of deviation in air pressure over the time. For the auditory system, the most simplistic type of sound wave is a sinusoidal variation in the air pressure. This kind of sound wave activates the hairs in the cochlea hence that nerve impulses contribute to a single neural pathway (figure 5.2). A sinusoidal sound wave can be characterized by a function of time.

$$q(t) = A_0 + A \sin(\omega t - \delta) \qquad (1)$$

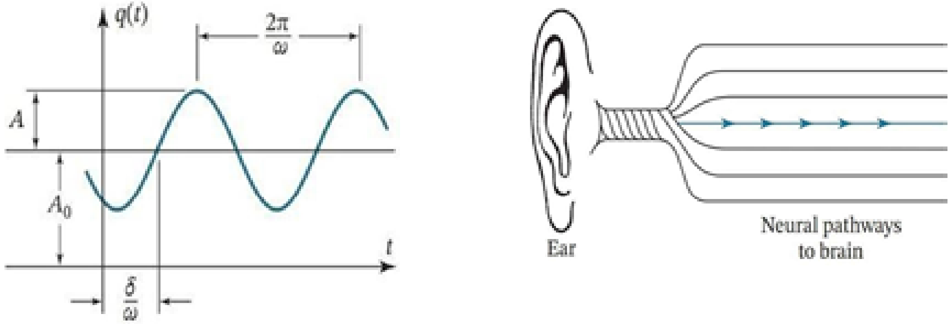

Fig 5.2

Here, q(t) is the atmospheric pressure at the eardrum, A_0 is the normal atmospheric pressure, A is the maximum deviation of pressure from the normal atmospheric pressure, *w*/2*p* is the frequency of the wave in cycles per unit time and d is the phase angle of the wave. To be interpreted as sound, such sinusoidal waves should have frequencies under a particular range. For human beings this range is approximately 20 Hz to 20,000 Hz. The frequencies beyond this range will not stimulate the hairs inside the cochlea enough to compose the nerve signals.

To the practicable intensity of perfection, the ear is a linear system. This points to that if a complex sound wave is a finite sum of sinusoidal components of at variance amplitudes, frequencies and phase angles, claim;

$$q(t)= A_0+A_1\sin(\omega_1 t- \delta_1) + A_2\sin(\omega_2 t- \delta_2)+ \ldots\ldots+ A_n\sin(\omega_n t- \delta_n) \qquad (2)$$

the response of the ear is made up of nerve impulses that are activated by the individual components, and those go through the same neural pathways (Fig 5.3).

Now, we consider some periodic sound wave p(t) with periodic T. Specifically, it is not a finite sum of sinusoidal waves. If we check out the response of the ear to such a periodic wave, we observe that it is the similar as the response to certain number of wave that is the complex of sinusoidal waves. Hence, there is few sound wave q(t) as given by Equation (2), that produces

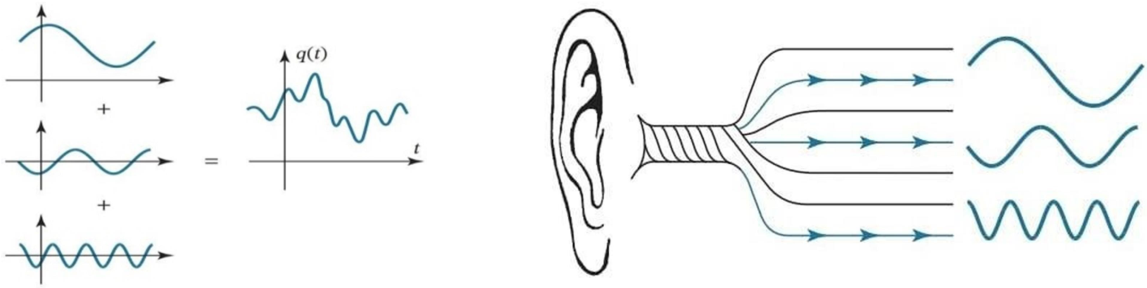

Fig 5.3

the similar reaction as p(t), even if p(t) and q(t) are different functions of time. [i.e., p(t) ≡ p (t + T)].

Here, we want to find out the frequencies, amplitudes and phase angles of the sinusoidal components of q(t). We can logical to expect that q(t) and p(t) have the same period T, since q(t) result in the same response as the periodic wave p(t), This preferred that the each sinusoidal term in q(t) have period T. As a result, the frequencies of the sinusoidal components should be integer multiple of the frequency 1/T of the function p(t).Then, the ω_κ in Equation(2) need to be of the form; $\omega\kappa = 2k\pi/T$, k = 1, 2, ... whereas the ear cannot recognize sinusoidal waves with frequencies more than 20,000 Hz, if we get values higher than 20,000 Hz, we may exclude these values of k for which $\omega\kappa/2\pi = k/T$. Consequently, q(t) is of the form;

$$q(t)= A_0+A_1\sin\left(\frac{2\pi t}{T} - \delta_1\right) + A_2\sin\left(\frac{4\pi t}{T} - \delta_2\right)+ ....+ A_n\sin\left(\frac{2n\pi t}{T} - \delta_n\right) \quad (3)$$

Here, the greatest integer n such that n/T doesn't exceed 20,000.

Afterwards we want to concentrate on the values of the amplitudes $A_0, A_1,\ldots, A_n$ and Phase angles $\delta_1, \delta_2,\ldots,\delta_n$ that are show in Equation (3). The auditory system "picks" these values according to a certain criterion such that q(t) leads to the same response as p(t).

For the examination of this criterion, we assume that, e(t) = p(t) “ q(t). If we suppose that q(t) is approximated to p(t), then the error of this approximation is denoted as e(t). The ear is unable to detect the errors. with reference to e(t), if the quantity is least as probable, then we can establish the amplitude and phase angles based on those criteria.

$$\int_0^T \left[e(t)\right]^2 dt = \int_0^T \left[p(t) - q(t)\right]^2 dt \quad (4)$$

We consider, the physiological aspects of the human auditory system, which consist of a fluid filled portion with millions of tiny hairs named cochlea within the inner ear. When the sound waves enter the cochlea, the hair cells start to vibrate and also cause the movement of molecules in the fluid there. Consequently, there generate an energy called acoustic energy. We note this energy is proportional to the error wave e(t). That is, acoustic energy is accurately calculated based on the difference in error between the observed sound wave p(t) and the corresponding estimated sound wave q(t),which in the case of the human ear, can distinguish between those waves. When the two waves provide the similar perception of sound, then there is the energy is least as possible. Algebraically, consider the vector space C [0,T] and the continuous functions on the interval [0, T], then the function p(t) in (4) is the least squares approximation by the q(t). The integral in (4) is called the least squares error, or the mean squares error, of the approximation.

5.3 General Theory

We develop the mathematical theory of the least squares approximation for a function with the linear combination of sinusoidal functions. Let f(t) be a given continuous function defined over an interval of the t-axis. We first consider the case when the interval is [0, T] for arbitrary T. Analogous to Eq (3) with T= 2π, we desire to approximate f(t) by a function of the form;

$$\text{g(t)} = A_0 + A_1 sin\left(t-\delta_1\right) + A_2 sin2\left(t-\delta_2\right) + \ldots + A_n sinn(t-\delta_n)$$

For some fixed integer n. Since sin k $(t - \delta_\kappa)$ is expressible as a linear combination of sin kt and cos kt, we can write g(t) in the alternate form;

$$\text{g(t)} = a_0 + a_1 cost + a_2 cos2t + .. + a_n cosnt + b_1 sint + b_2 sin2t + .. + b_n sinnt \tag{5}$$

Such a function is called a trigonometric polynomial of order n. Our problem is to find values of a_0, a_1, ..., a_n, b_1, ..., b_n such that g(t) is the least squares approximation to f(t) on the interval [0, 2π]. That is, the coefficients are to be chosen so that the least squares error is as least as possible.

$$\int_0^{2\pi} \left[f\left(t\right)-g\left(t\right)\right]^2 dt \tag{6}$$

Since the integral in Eq.(6) is a function of the 2n+1 coefficients a_0, a_1, ..., a_n, b_1, ..., b_n it is possible to use calculus to find the minimum value of the least squares error and the corresponding values of these 2n+1 coefficients. However, an approach using Linear Algebra will give us greater insight into the nature of the approximation process. Moreover, the method we discuss can be applied to many least square problems besides those in this thesis. We will need the following three facts:

1. The function f(t) we are attempting to approximate may be viewed as a vector in the vector space C [0, 2π] consist of continuous functions in the interval of [0, 2π].

2. Since the approximating function g(t) is a linear combination of 1, cos t, …… cos nt, sin t ..., sin nt, we may view g(t) as a vector in the subspace W of C [0, 2π] spanned by these 2n+1 vectors.
3. Since

$$\|f-g\|=\sqrt{\int_0^{2\pi}\left[f(t)-g(t)\right]^2 dt} \qquad (7)$$

is the distance between $f(t)$ and $g(t)$ in the norm generated by the inner product.

$$<u,v>=\int_0^{2\pi}u(t)v(t)\,dt \qquad (8)$$

the least squares error

$$\int_0^{2\pi}\left[f(t)-g(t)\right]^2 dt \qquad (9)$$

Represents the square of the distance ||f – g||.

In light of these remarks, the problem of finding a trigonometric polynomial g(t) which minimizes the least squares error given by (9) is equivalent to the problem of finding a vector g in the subspace W which minimizes the distance ||f – g||. The latter problem can be solved by use of the following theorem from the theory of inner product spaces (Fig 5.4).

THEOREM 1

Let f be vector belongs to an inner product space and let W be a finite dimensional subspace. Then the vector g in W which minimizes the distance ||f – g|| is $proj_w f$, the orthogonal projection of onto W. If the vectors $g_o, g_1 .... g_m$ forms an orthonormal basis for W,

$$Proj_w f = <f,g_0>g_0 + <f,g_1>g_1 + \ldots + <f,g_m>g_m \qquad (10)$$

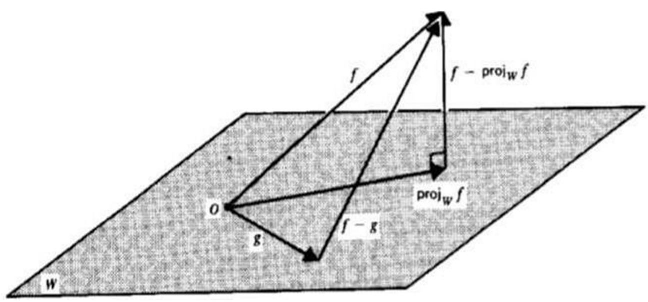

Fig 5.4

To apply this theorem, we must first find an orthonormal basis for the subspace W spanned by the 2n+1 vectors 1, cos t, cos 2t,..., cos nt, sint, sin 2t,..., sin nt. A direct calculation verifies that these 2n+1 vectors are orthogonal relative to the inner product. Consequently, we need only divide each one of these vectors by its length to generate an orthonormal basis for W. The result is;

$$g_0 = \frac{1}{\sqrt{2\pi}}, g_1 = \frac{1}{\sqrt{\pi}} cost, \ldots g_n = \frac{1}{\sqrt{\pi}} cosnt, g_{n+1} = \frac{1}{\sqrt{\pi}} sinnt, g_{2n} = \frac{1}{\sqrt{\pi}} sinnt$$

The orthogonal projection of f onto W is given by

$$proj_w f = < f, g_0 > \frac{1}{\sqrt{2\pi}} + < f, g_1 > \frac{1}{\sqrt{\pi}} cost + \ldots + < f, g_n > \frac{1}{\sqrt{\pi}} cosnt + < f, g_{n+1} > \frac{1}{\sqrt{\pi}} sint + \ldots + < f, g_{2n} > \frac{1}{\sqrt{\pi}} sinnt \tag{11}$$

To simplify our notation, let us define

$$a_0 = 2 < f, g_0 > \frac{1}{\sqrt{2\pi}} = \frac{1}{\pi} \int_0^{2\pi} f(t)\,dt$$

$$a_k = < f, g_k > \frac{1}{\sqrt{\pi}} = \frac{1}{\pi} \int_0^{2\pi} f(t) cosktdt \qquad k = 1,2,3,\ldots,n$$

$$b_k = < f, g_{n+k} > \frac{1}{\sqrt{\pi}} = \frac{1}{\pi} \int_0^{2\pi} f(t) sinktdt \qquad k = 1,2,3,\ldots,n$$

Equation (11) can then be written as

$$proj_w f = \frac{1}{2} a_0 + a_1 cost + \ldots + a_n cosnt + b_1 sint + \ldots + b_n sinnt$$

In summary, we have the following result :

THEOREM 2

If f(t) is continuous on [0,2π], the trigonometric function g(t) of the form:

$$g(t) = \frac{1}{2} a_0 + a_1 cost + \ldots + a_n cosnt + b_1 sint + \ldots + b_n sinnt$$

Which minimizes the least squares error $\int_0^{2\pi} \left[f(t) - g(t) \right]^2 dt$ has coefficients

$$a_k = \frac{1}{\pi} \int_0^{2\pi} f(t) cosktdt, \qquad k = 0,1,2,\ldots,n$$

$$b_k = \frac{1}{\pi}\int_0^{2\pi} f(t)\, sinktdt, \qquad k = 1, 2, 3, \ldots, n$$

If the function f(t) is defined on the interval [0, T] in place of [0, 2π], a change of dimensions will given the following result.

THEOREM 3

If f(t) is continuous on [0, T] , then the trigonometric function g(t) of the form:

$$g(t) = \frac{1}{2}a_0 + a_1\cos\left(\frac{2\pi t}{T}\right) + \ldots + a_n \cos\left(\frac{2n\pi t}{T}\right) + b_1 sin\left(\frac{2\pi t}{T}\right) + \ldots + b_n sin(\frac{2n\pi t}{T})$$

This minimizes the least squares error $\int_0^T \left[f(t) - g(t)\right]^2 dt$ has coefficients

$$a_k = \frac{2}{T}\int_0^T f(t)\cos\left(\frac{2k\pi t}{T}\right)dt \qquad k = 0, 1, 2, \ldots.., n$$

$$b_k = \frac{2}{T}\int_0^T f(t)\sin\left(\frac{2k\pi t}{T}\right)dt \qquad k = 1, 2, 3\ldots, n$$

5.3 Least Square Approximation

Consider a sound wave p(t) with a basic frequency of 5000 Hz which have a saw-tooth pattern. Suppose units have been pick out so that the normal atmospheric pressure is at the zero level, and the maximum deviation of amplitude of the wave is A. The fundamental period of the wave is T=1/5000 = 0.0002 seconds. From t=0 to t=T, the function p(t) has the equation;

$$p(t) = \frac{2A}{T}\left(\frac{T}{2} - t\right)$$

Then by the theorem 3, given the following :

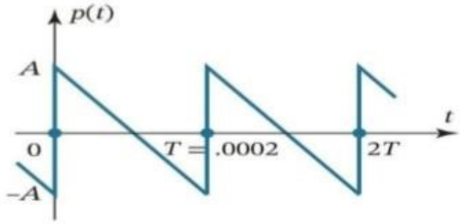

Fig 5.5

$$a_0 = \frac{2}{T}\int_0^T p(t)\,dt = \frac{2}{T}\int_0^T \frac{2A}{T}\left(\frac{T}{2} - t\right)dt = 0$$

$$a_k = \frac{2}{T}\int_0^T p(t)\cos\left(\frac{2k\pi t}{T}\right)dt = \frac{2}{T}\int_0^T \frac{2A}{T}\left(\frac{T}{2} - t\right)\cos\left(\frac{2k\pi t}{T}\right)dt = 0, \quad k = 1, 2 \ldots\ldots$$

$$b_k = \frac{2}{T}\int_0^T p(t)\sin\left(\frac{2k\pi t}{T}\right)dt = \frac{2}{T}\int_0^T \frac{2A}{T}\left(\frac{T}{2} - t\right)\sin\left(\frac{2k\pi t}{T}\right)dt = \frac{2A}{k\pi}, \quad k = 1, 2 \ldots\ldots\ldots$$

Let's evaluate, how the sound wave p(t) is identified by the human ear. We observe that 4/T 20,000 Hz, in this case we want only move up to k=4 in the above expressions. The least squares approximation to p(t) is then,

$$q(t) = \frac{2A}{\pi}\left[sin\left(\frac{2\pi}{T}\right) + \frac{1}{2} sin\left(\frac{4\pi t}{T}\right) + \frac{1}{3} sin\left(\frac{6\pi t}{T}\right) + \frac{1}{4} sin\left(\frac{8\pi t}{T}\right)\right]$$

The four sinusoidal components have frequencies of 5,000, 10,000, 15,000 and 20,000 Hz, respectively. In Fig 5.6, we represent the p(t) and q(t) over one period. Here, we obtain the q(t) is not an accurate point-by-point approximation to p(t). However, to the human ear this two waves p(t) and q(t) produce the same sensation of sound.

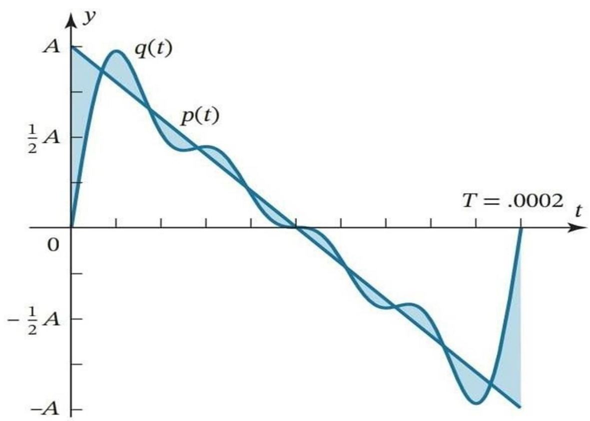

Fig 5.6

Obviously, if the number of components in the approximating trigonometric polynomial increases, then the least squares approximation enhances the result. In more advanced courses, it is shown that the least squares error tends to zero as n approaches infinity. For a function f(t) defined over the interval [0, 2π], this limiting approximation is denoted by

$$f(t) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty}(a_k coskt + b_k sinkt) \quad (12)$$

and is called the Fourier series of f(t) on the interval [0, 2π]. The equality in this equation denotes equality between the two sides of the equation considered as vectors in C [0, 2π]. To be precise, Eq. (12) denotes the fact that the quantity tends to zero as n tends to infinity. Even if the Fourier series of f(t) converges to f(t) for each value of t.

$$\int_0^{2\pi}[f(t) - \frac{1}{2}a_0 - \sum_{k=1}^{n}(a_k coskt + b_k sinkt)]^2\, dt$$

Finally, we formulated a least square model for describing the perceived sound wave by human hearing. The least squares model is a mathematical model in which we applied the general theory of least square approximation to distinguish sound waves.

6. CONCLUSION

The mathematical modelling formulation for human hearing is one of the great innovations to the future world. Here, we take the first move towards the mathematical model formulation for human hearing.

Firstly, we analyzed the evolution of human hearing, history and physiological studies related to the hearing mechanism of human beings. The second chapter also gave the detailed explanation of sound conduction through the outer ear and middle ear as part of hearing mechanism. Through the third chapter, we explained the transduction of sound waves in the inner ear for hearing mechanism. Among the fourth chapter, we precisely illustrate about the great part of outer ear, middle ear and inner ear in human auditory system.

Finally, we formulated the mathematical model for human hearing. The model was analyzed and solved by the general theory of least square approximation and is motivated by energy consideration. That is for analysing, we take the periodic wave p(t) with period T, as expansion of Fourier series and is approximated to the complex sound wave q(t). We approach the linear algebraic theory for the least square approximation, finally we got the response of the ear to p (t) is the same as the response of the ear to q (t). That is, these two waves produce the same sensation of sound.

The limitations encountered in this paper are non-availability of resource materials; proper power supply and lack of experimental material such as human ear, wave meter and digital multimeter can measure the frequency. and digital multimeter can measure the frequency.

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